

# HAMILTONIANS ARISING FROM L-FUNCTIONS IN THE SELBERG CLASS

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ABSTRACT. We establish a new equivalent condition for the Grand Riemann Hypothesis for  $L$ -functions in a wide subclass of the Selberg class in terms of canonical systems of differential equations. A canonical system is determined by a real symmetric matrix valued function called a Hamiltonian. To establish the equivalent condition, we solve and use an inverse spectral problem for canonical systems of special type.

## 1. INTRODUCTION

The Riemann Hypothesis (RH) asserts that all nontrivial zeros of the Riemann zeta-function  $\zeta(s)$  lie on the critical line  $\Re(s) = 1/2$ , and it had been generalized to wider classes of zeta-like functions. Especially, the analogue of RH for  $L$ -functions in the Selberg class is often called the Grand Riemann Hypothesis (GRH)<sup>1</sup>.

Briefly, we have two issues in this paper. The first is the resolution of an inverse spectral problem for canonical systems of differential equations. The second is the establishment of a new equivalent condition for GRH for  $L$ -functions in a wide subclass of the Selberg class in terms of canonical systems. These two issues are closely related with each other. We explain the relation by dealing with the case of the Riemann zeta function as the introduction. The present study was mainly stimulated by the works of J. C. Lagarias [25, 26, 27] and J.-F. Burnol [10, 11, 12, 13].

The Riemann xi-function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

is an entire function taking real-values on the critical line such that the zeros coincide with nontrivial zeros of  $\zeta(s)$ . Therefore RH is equivalent that all zeros of the entire function  $\xi(s)$  lie on the critical line. Noting this, we start from the consideration on the additive decomposition

$$\xi\left(\frac{1}{2} - iz\right) = \frac{1}{2}(E(z) + E^\sharp(z)) \quad (1.1)$$

by an entire function  $E$ , where  $i = \sqrt{-1}$ ,  $F^\sharp(z) = \overline{F(\bar{z})}$  for an entire function  $F$  and the bar stands for the complex conjugate. It is easily confirmed that (1.1) holds for

$$E_\xi(z) = \xi\left(\frac{1}{2} - iz\right) + \xi'\left(\frac{1}{2} - iz\right) \quad (1.2)$$

by the functional equations  $\xi(s) = \xi(1-s)$  and  $\xi(s) = \xi^\sharp(s)$  and thus it also holds for infinitely many entire functions  $E = E_\xi + F$ , where the prime stands for the derivative with respect to  $s$  and  $F$  is an entire function satisfying  $F^\sharp = -F$ .

The advantage of the decomposition (1.1) stands on the theory of the *Hermite–Biehler class* of entire functions. We denote by  $\mathbb{HB}$  the set of all entire functions satisfying

$$|E^\sharp(z)| < |E(z)| \quad \text{for all } z \in \mathbb{C}_+, \quad (1.3)$$

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<sup>1</sup> The abbreviation GRH is used often to the Generalized Riemann Hypothesis in literatures but we use this to the Grand Riemann Hypothesis throughout this paper.

where  $\mathbb{C}_+ = \{z = u + iv \mid u, v \in \mathbb{R}, v > 0\}$  is the upper half-plane. The Hermite–Biehler class  $\mathbb{H}\mathbb{B}$  consists of all  $E \in \overline{\mathbb{H}\mathbb{B}}$  having no real zeros<sup>2</sup>. An entire function  $F$  satisfying  $F = F^\sharp$  is called a real entire function. In general,  $E \in \overline{\mathbb{H}\mathbb{B}}$  implies that two real entire functions

$$A(z) := \frac{1}{2}(E(z) + E^\sharp(z)) \quad \text{and} \quad B(z) := \frac{i}{2}(E(z) - E^\sharp(z)) \quad (1.4)$$

have only real zeros, and these zeros interlace. If  $E \in \mathbb{H}\mathbb{B}$ , all (real) zeros of  $A$  and  $B$  are simple ([4]). Therefore, the existence of  $E \in \mathbb{H}\mathbb{B}$  satisfying (1.1) implies that RH holds together with the Simplicity Conjecture (SC) which assert that all nontrivial zeros of  $\zeta(s)$  are simple. Conversely, there exists  $E \in \mathbb{H}\mathbb{B}$  satisfying (1.1) if we assume that RH and SC hold. In fact,  $E_\xi$  of (1.2) belongs to  $\mathbb{H}\mathbb{B}$  under RH and SC ([26]).

The above discussion suggest the following strategy to the proof of RH with SC: first, find an entire function  $E$  satisfying (1.1); second, prove that  $E$  belongs to  $\mathbb{H}\mathbb{B}$ . Then these two conditions conclude RH and SC as the above. More simply, we may start from the second step by using  $E_\xi$  of (1.2). The obvious difficulty of the above strategy is the second step, in other words, we do not know any reason why an entire function  $E$  satisfying (1.1) (or  $E_\xi$ ) should belong to  $\mathbb{H}\mathbb{B}$  if RH holds. In this paper, we search for the reason of  $E \in \mathbb{H}\mathbb{B}$  in the theory of de Branges on canonical systems as with Lagarias [25, 26, 27] in which the applicability of the theory of de Branges to the study of the zeros of  $L$ -functions is suggested.

To start with, we explain that canonical systems generate functions of  $\overline{\mathbb{H}\mathbb{B}}$ . Let  $\text{Sym}_2(\mathbb{R})$  be the set of all  $2 \times 2$  real symmetric matrices. A  $\text{Sym}_2(\mathbb{R})$ -valued function  $H$  defined on  $I = [t_1, t_0)$  ( $-\infty < t_1 < t_0 \leq \infty$ ) is called a *Hamiltonian* if

- (H1)  $H(t)$  is positive semidefinite for almost every  $t \in I$ ,
- (H2)  $H \not\equiv 0$  on any subset of  $I$  with positive Lebesgue measure,
- (H3)  $H$  is locally integrable on  $I$  with respect to the Lebesgue measure.

An open subinterval  $J$  of  $I$  is called *H-indivisible*, if the equality

$$H(t) = h(t) \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix}$$

holds on  $J$  for some positive function  $h$  on  $J$  and  $0 \leq \theta < \pi$ . A point  $t \in I$  is called *regular* if it does not belong to any  $H$ -indivisible interval, otherwise  $t$  is called *singular*.

The first-order system

$$-\frac{d}{dt} \begin{bmatrix} A(t, z) \\ B(t, z) \end{bmatrix} = z \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} H(t) \begin{bmatrix} A(t, z) \\ B(t, z) \end{bmatrix}, \quad z \in \mathbb{C} \quad (1.5)$$

associated with a  $\text{Sym}_2(\mathbb{R})$ -valued function  $H$  on  $I$  is called a *canonical system* on  $I$  if  $H$  is a Hamiltonian. See the survey articles H. Winkler [45], H. Woracek [46] and references there in for theoretical and historical details on canonical systems (of dimension two).

If  ${}^t(A(t, z), B(t, z))$  is a solution of a canonical system on  $I = [t_1, t_0)$  such that  $\lim_{t \rightarrow t_0} J(t; z, z) = 0$  for all  $z \in \mathbb{C}_+$ , the function  $E(t, z) = A(t, z) - iB(t, z)$  is an entire function of  $\overline{\mathbb{H}\mathbb{B}}$  for every regular  $t \in I$  (see Proposition 3.4 below for a special  $H$ ). To derive more strong conclusion  $E(t, z) \in \mathbb{H}\mathbb{B}$ , we need information at  $t = t_0$  in general. For instance,  $E(t, z) \in \mathbb{H}\mathbb{B}$  for every regular  $t \in I$  if  $(A(t, z), B(t, z))$  tends to  $(c, 0)$  as  $t \rightarrow t_0$  for some constant  $c \neq 0$ .

<sup>2</sup> The definitions of  $\mathbb{H}\mathbb{B}$  is equivalent to the definition of Levin [28, §1 and §2 of Chap. VII] if we replace the word “the upper half-plane” by “the lower half-plane”, because (1.3) implies that  $E(z)$  has no zeros in  $\mathbb{C}_+$ . On the other hand, note that  $\overline{\mathbb{H}\mathbb{B}}$  of this paper is different from  $\overline{\mathbb{H}\mathbb{B}}$  of [28]. We adopt the above definition for the convenience to use the theory of canonical systems. Note that a member of  $\mathbb{H}\mathbb{B}$  is called a strict structure function in [26] and that a member of  $\overline{\mathbb{H}\mathbb{B}}$  is called a de Brange structure function and a structure function in [25] and [26, 27], respectively.

A fundamental and quite important result is the resolution of the inverse spectral problem that recovers  $H$  from  $E \in \mathbb{H}\mathbb{B}$ . This inverse spectral problem had been solved by L. de Branges as follows ([9, Theorem 40], see also [7, Theorem II]):

**Theorem dB.** *For every  $E \in \mathbb{H}\mathbb{B}$ , there exists a Hamiltonian  $H$  on some (possibly unbounded) interval  $I = [t_1, t_0)$  having the following properties:*

- (1) *For the unique solution  ${}^t(A(t, z), B(t, z))$  of the initial value problem (1.5) with*

$$(A(t_1, z), B(t_1, z)) = (A(z), B(z)),$$

*$E(t, z) := A(t, z) - iB(t, z)$  belongs to  $\mathbb{H}\mathbb{B}$  for every regular  $t \in I$ .*

- (2) *Define*

$$J(t; z, w) := \frac{\overline{A(t, z)}B(t, w) - A(t, w)\overline{B(t, z)}}{\pi(w - \bar{z})}. \quad (1.6)$$

*for  $(A(t, z), B(t, z))$  of (1). Then  $\lim_{t \rightarrow t_0} J(t; z, z) = 0$  for all  $z \in \mathbb{C}$ .*

We abbreviate to  $\text{Inv}(E)$  the triple  $(I, H(t), (A(t, z), B(t, z)))$  in Theorem dB. The triple  $\text{Inv}(E)$  is unique up to the scaling of  $t$  under the normalization  $E(0) = 1$  and  $\text{tr } H(t) = 1$ , but we do not require such normalizations. If  $(I, H(t), (A(t, z), B(t, z)))$  satisfies the conditions in Theorem dB, the triple  $(J, H(t), (A(t, z), B(t, z)))$  for a subinterval  $J \subset I$  satisfies the conditions in Theorem dB without (2). We abbreviate to  $\text{Inv}(E)^b$  such a triple. The only difference between  $\text{Inv}(E)$  and  $\text{Inv}(E)^b$  is the domain of  $H$ .

By Theorem dB, if we assume that RH and SC hold, there exists a Hamiltonian  $H_\xi$  defined on some interval  $I$  such that  $E_\xi$  of (1.2) is recovered from the solution of the canonical system associated with  $H_\xi$  by applying de Branges' result to  $E_\xi$ . Using the conjectural Hamiltonian  $H_\xi$ , the above naive strategy to the proof of RH is now refined as follows:

- (1-1) Constructing the Hamiltonian  $H_\xi$  on  $I = [t_1, t_0)$  without RH;
- (1-2) Constructing the solution  $(A_\xi(t, z), B_\xi(t, z))$  of the canonical system on  $I$  associated with  $H_\xi$  satisfying  $E_\xi(z) = A_\xi(t_1, z) - iB_\xi(t_1, z)$ ;
- (1-3) Showing that  $\lim_{t \rightarrow t_0} J_\xi(t; z, z) = 0$  for all  $z \in \mathbb{C}_+$ , where  $J_\xi(t; z, w)$  is the function defined by (1.6) for  $(A_\xi(t, z), B_\xi(t, z))$ .

It is concluded that  $E_\xi$  belongs to  $\overline{\mathbb{H}\mathbb{B}}$  if the above three steps are completed (cf. Proposition 3.4). Therefore, all zeros of  $\xi(\frac{1}{2} - iz) = A_\xi(z) = \frac{1}{2}(E_\xi(z) + E_\xi^\sharp(z))$  are real, this is nothing but the Riemann hypothesis. In this approach, we face a serious obstacle from the first step. That is, to carry out it, we need an explicit formula of  $H_\xi$  obtained from  $E_\xi$  under RH. However, the explicit construction of  $H$  for given  $E \in \mathbb{H}\mathbb{B}$  is difficult in general as well as the other inverse spectral problem, and it is usually not possible to obtain enough analytic information of  $H$  in known constructions. In fact, an explicit form of  $H$  is not known except for a few examples of  $E$  as in [9, Chapter 3], [16, Section 8] (and also [13], [27]) and some additional examples constructed from such known examples using transformation rules for Hamiltonians ([44]). More seriously, known constructions of  $H$  is not applicable to  $E$  if it is not known whether  $E \in \mathbb{H}\mathbb{B}$ .

In order to avoid the above obstacles, we consider the family of entire functions

$$E_\xi^\omega(z) := \xi(\tfrac{1}{2} + \omega - iz), \quad \omega > 0$$

instead of the single function  $E_\xi$ . (Note that  $E_\xi \neq E_\xi^0$ , but  $E_\xi^\omega$  for small  $\omega > 0$  is similar to  $E_\xi$  in the sense that  $E_\xi^\omega(z) = \xi(\frac{1}{2} + \omega - iz) + \omega \xi'(\frac{1}{2} + \omega - iz) + O(\omega^2)$  for small  $\omega > 0$  if  $z$  in a compact set.) Then we find that a necessary and sufficient condition for RH (not require SC) is that  $E_\xi^\omega \in \mathbb{H}\mathbb{B}$  for every  $\omega > 0$  (Propositions 2.1 and 2.2). In particular, there exists a Hamiltonian  $H_\xi^\omega$  for every  $\omega > 0$  under RH. Therefore, RH is proved by completing the following three steps for every  $\omega > 0$ :

- (2-1) Constructing the Hamiltonian  $H_\xi^\omega$  on  $I = [t_1, t_0)$  without RH;
- (2-2) Constructing the solution  $(A_\xi^\omega(t, z), B_\xi^\omega(t, z))$  of the canonical system on  $I$  associated with  $H_\xi^\omega$  satisfying  $E_\xi^\omega(z) = A_\xi^\omega(t_1, z) - iB_\xi^\omega(t_1, z)$ ;
- (2-3) Showing that  $\lim_{t \rightarrow t_0} J_\xi^\omega(t; z, z) = 0$  for all  $z \in \mathbb{C}_+$ .

There are two advantages of the second strategy. The first advantage is that the Hamiltonian  $H_\xi^\omega$  on  $I = [0, \infty)$  and the solution  $(A_\xi^\omega(t, z), B_\xi^\omega(t, z))$  of the associate canonical system are explicitly constructed in [40] under the restriction  $\omega > 1$  by applying the method of Burnol [13] introduced for the study of the Hankel transform. The second advantage is the avoiding of SC and the central zero, that is, multiple zeros on the critical line and the zero at the central point  $s = 1/2$  are allowed to  $\xi$  in the second strategy. This point is important to generalize the above strategy to the other  $L$ -functions, because they often have a multiple zero at the central point  $s = 1/2$ .

In [40],  $H_\xi^\omega$  and  $(A_\xi^\omega(t, z), B_\xi^\omega(t, z))$  are constructed by using solutions  $\varphi_t^\pm$  of the integral equations

$$\varphi_t^\pm(x) \pm \int_{-\infty}^t K_\xi^\omega(x+y) \varphi_t^\pm(y) dy = K_\xi^\omega(x+t),$$

where  $K_\xi^\omega$  is the kernel defined by

$$K_\xi^\omega(x) = \frac{1}{2\pi} \int_{\Im(z)=c} \frac{(E_\xi^\omega)^\sharp(z)}{E_\xi^\omega(z)} e^{-izx} dz$$

for large  $c > 0$ . The behavior of  $(A_\xi^\omega(t, z), B_\xi^\omega(t, z))$  at  $t = \infty$  and its role in the proof of  $E_\xi^\omega \in \mathbb{H}\mathbb{B}$  were not studied in [40]. However, if the construction of  $H_\xi^\omega$  and  $(A_\xi^\omega(t, z), B_\xi^\omega(t, z))$  is extended to  $0 < \omega \leq 1$  together with an additional result on the behavior of  $(A_\xi^\omega(t, z), B_\xi^\omega(t, z))$  at  $t = \infty$ , we obtain  $E_\xi^\omega \in \mathbb{H}\mathbb{B}$  for every  $\omega > 0$ , which implies RH. Unfortunately, there were several technical difficulties in [40] to extend the construction of  $H_\xi^\omega$  and  $(A_\xi^\omega(t, z), B_\xi^\omega(t, z))$  to  $0 < \omega \leq 1$ . For instance, the above  $K_L^\omega$  is far from continuous functions and  $L^2$ -functions if  $\omega > 0$  is small, and this fact is a serious obstacle for the construction in [40].

In this paper, we resolve the above technical difficulties by introducing the additional discrete parameter  $\nu$ :

$$E_\xi^{\omega, \nu}(z) = \xi(\tfrac{1}{2} + \omega - iz)^\nu, \quad \omega \in \mathbb{R}_{>0}, \quad \nu \in \mathbb{Z}_{>0}.$$

The parameter  $\nu$  does not affect to study whether  $E_L^{\omega, \nu} \in \mathbb{H}\mathbb{B}$  by definition of  $\mathbb{H}\mathbb{B}$ , but it plays an important role in the construction of  $\text{Inv}(E_\xi^{\omega, \nu})^\flat$ . The conjectural triple  $\text{Inv}(E_\xi^{\omega, \nu})$  will be obtained by applying a general construction of  $\text{Inv}(E)^\flat$  in Section 3 for entire functions  $E$  satisfying several conditions. We will find in Section 4 that  $E_L^{\omega, \nu}$  satisfies such conditions owing to the parameter  $\nu$ . In this way, a large part of (2-1) and (2-2) are achieved successfully for each  $\omega > 0$ .

Summarizing the above discussion, we obtain an equivalent condition for RH in terms of canonical systems associated with Hamiltonians. This framework to establish the equivalent condition of RH is applicable to more general zeta- and  $L$ -functions. We apply it to  $L$ -functions in the Selberg class which was introduced in Selberg [36] together with a sophisticated consideration about the question what is an  $L$ -function. Then we obtain an equivalent condition for the analogue of RH for  $L$ -functions in the Selberg class as in Section 2. This is the goal of this paper.

Before concluding the introduction, we comment on the Hilbert-Pólya conjecture, a conjectural possible approach to RH. It claims that the imaginary parts of the nontrivial zeros of  $\zeta(s)$  are eigenvalues of some unbounded self-adjoint operator  $D$  acting on a Hilbert space  $\mathcal{H}$ . The Montgomery-Odlyzko conjecture on the vertical distribution of

the nontrivial zeros of  $\zeta(s)$  and the resemblance between the Weil explicit formula and the Selberg trace formula are strong evidences to the Hilbert-Pólya conjecture. No pair  $(\mathcal{H}, D)$  of space and operator had been found, although the conjectural pair were suggested by several authors. Among them, the idea of A. Connes [14] for the conjectural Hilbert-Pólya pair  $(\mathcal{H}, D)$  is very attractive in the sense that it stands on the local-global principle in number theory (adeles and ideles), it enables us to understand the Weil explicit formula as a trace formula, and it is stated not only for  $\zeta(s)$  but also Dedekind zeta-functions and Hecke  $L$ -functions. In addition, his idea is compatible with the Berry-Keating model [2] which is an attempt to give an explanation for RH by using physical model.

If  $E_\xi^{\omega, \nu} \in \mathbb{H}\mathbb{B}$ , we can construct the family of pairs  $\{(\mathcal{H}^{\omega, \nu(\omega)}, D^{\omega, \nu(\omega)})\}_{\omega > 0}$  of Hilbert spaces and self-adjoint operators. This family may be regarded as a possible realization of Connes' Hilbert-Pólya pair  $(\mathcal{H}, D)$  by allowing the perturbation parameter  $\omega$ . This topic will be treated more precisely in Section 8.

The paper is organized as follows. In Section 2, we state the main results Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4 after a small preparation of notation. The first two theorems are used to associate the theory of canonical systems to GRH via the inverse spectral problem  $\text{Inv}(E)^b$ . The third theorem is related with the necessity of GRH in terms of  $\text{Inv}(E)$ . The fourth theorem is the goal of this paper which is an equivalent condition for GRH in terms of  $\text{Inv}(E)$ . In Section 3, we solve the inverse spectral problem  $\text{Inv}(E)^b$  in Theorem 3.1 by assuming five conditions to an entire function  $E$ . The way of construction of  $\text{Inv}(E)^b$  is similar to [13] and [40]. In Section 4, we prove that  $E_\xi^{\omega, \nu}$  and its generalization  $E_L^{\omega, \nu}$  to  $L$ -functions in the Selberg class satisfy the first four conditions assumed in Theorem 3.1. Successively, we prove Theorem 2.1 which assert that  $E_L^{\omega, \nu}$  satisfies the fifth condition assumed in Theorem 3.1. Then, we obtain Theorem 2.2 by applying Theorem 3.1 to  $E_L^{\omega, \nu}$ . In Section 5, we review the theory of de Branges spaces as preparation for the proof of Theorem 2.3. In Section 6, we prove Theorem 2.3 by studying subspaces of certain de Branges spaces. In Section 7, we prove Theorem 2.4 by combining several results in former sections. In Section 8, we comment on the Hilbert-Pólya conjecture and Connes' approach about it from the viewpoint of the theory of canonical systems. In Section 9, we state and prove several complementary or digressive results. In Section 10, we give miscellaneous remarks on results and contents of this paper.

Basically, we attempt as much as possible to prove the main results in Section 2 by applying general results to  $L$ -functions in the Selberg class for the convenience of applications to other class of  $L$ -functions.

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## 2. MAIN RESULTS

**2.1. Selberg class and GRH.** Let  $s = \sigma + it$  be the complex variable. The Selberg class  $\mathcal{S}$  consists of the Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_L(n)}{n^s} \quad (2.1)$$

satisfying the following five axioms<sup>3</sup>:

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<sup>3</sup> As a matter of fact, (S1) is unnecessary to define  $\mathcal{S}$  because it is derived from (S4). But we put it into the axiom of  $\mathcal{S}$  according to other literature. A positive reason is that it is convenient to define the extended Selberg class which is the class of Dirichlet series satisfying (S1)~(S3).

- (S1) The Dirichlet series (2.1) converges absolutely if  $\sigma > 1$ .  
 (S2) **Analytic continuation** – There exists an integer  $m \geq 0$  such that  $(s-1)^m L(s)$  extends to an entire function of finite order.  
 (S3) **Functional equation** –  $L$  satisfies the functional equation

$$\Lambda_L(s) = \epsilon_L \Lambda_L^\sharp(1-s),$$

where

$$\Lambda_L(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j) \cdot L(s) = \gamma_L(s) \cdot L(s),$$

$\Gamma$  is the gamma function and  $r \geq 0$ ,  $Q > 0$ ,  $\lambda_j > 0$ ,  $\mu_j \in \mathbb{C}$  with  $\Re(\mu_j) \geq 0$ ,  $\epsilon_L \in \mathbb{C}$  with  $|\epsilon_L| = 1$  are parameters depending on  $L$ .

- (S4) **Ramanujan conjecture** – We have  $a_L(n) \ll_\varepsilon n^\varepsilon$  for every  $\varepsilon > 0$ .  
 (S5) **Euler product** – We have

$$\log L(s) = \sum_{n=1}^{\infty} \frac{b_L(n)}{n^s}$$

if  $\sigma > 1$ , where  $b_L(n) = 0$  unless  $n = p^m$  with  $m \geq 1$ , and  $b_L(n) \ll n^\theta$  for some  $\theta < 1/2$ .

The Riemann zeta-function and Dirichlet  $L$ -functions associated to primitive Dirichlet characters are typical members of the Selberg class  $\mathcal{S}$ . As with these examples, it is conjectured that all major zeta- and  $L$ -functions appearing in number theory, such as automorphic  $L$ -functions and Artin  $L$ -functions, are members of  $\mathcal{S}$ . Considering this conjecture,  $\mathcal{S}$  is a proper class of  $L$ -functions in studying the analogue of RH for number theoretic  $L$ -functions. See the survey article [35] of A. Perelli for an overview of results, conjectures and problems relating to the Selberg class.

We define the degree  $d_L$  of  $L \in \mathcal{S}$  by  $d_L = 2 \sum_{j=1}^r \lambda_j$ , where  $\lambda_j$  are numbers in (S3). From (S5), we have  $a_L(1) = 1$  and find that coefficients  $a_L(n)$  define a multiplicative arithmetic function. From (S3) and (S5),  $L \in \mathcal{S}$  has no zeros outside the critical strip  $0 \leq \sigma \leq 1$  except for zeros in the half-plane  $\sigma \leq 0$  located at poles of the involved gamma factors. The zeros lie in the critical strip are called the nontrivial zeros. The nontrivial zeros are infinitely many unless  $L \equiv 1$  and coincide with the zeros of the entire function<sup>4</sup>

$$\xi_L(s) = s^{m_L} (s-1)^{m_L} \Lambda_L(s) \quad (2.2)$$

of order one, where  $m_L$  is the minimal nonnegative integer  $m$  in (S2). It is conjectured that the analogue of RH holds for all  $L$ -functions in  $\mathcal{S}$ :

**Grand Riemann Hypothesis (GRH).** For  $L \in \mathcal{S}$ ,  $\xi_L(s) \neq 0$  unless  $\Re(s) = 1/2$ .

For  $L \in \mathcal{S}$ , we abbreviate to  $\text{GRH}(L)$  the assertion that all zeros of  $\xi_L(s)$  lie on the critical line  $\Re(s) = 1/2$ .

The main subject of this paper is the studying of  $\text{GRH}(L)$  for  $L$ -functions in the subclass  $\mathcal{S}_{\mathbb{R}}$  of  $\mathcal{S}$  defined by

$$\mathcal{S}_{\mathbb{R}} = \{L \in \mathcal{S} \mid L \not\equiv 1, L(\mathbb{R}) \subset \mathbb{R}, \Lambda_L(\mathbb{R}) \subset \mathbb{R}\}.$$

If  $L \in \mathcal{S}_{\mathbb{R}}$ , the gamma factor  $\gamma_L$  in (S3) is not an exponential function,  $\xi_L$  are non-constant entire functions satisfying functional equations

$$\xi_L(s) = \epsilon_L \xi_L(1-s) \quad \text{and} \quad \xi_L(s) = \xi_L^\sharp(s), \quad (2.3)$$

where the root number  $\epsilon_L$  must be 1 or  $-1$ . The second equality of (2.3) means that  $\xi_L$  is an real entire function.

<sup>4</sup> The quantity  $2 \sum_{j=1}^r (\mu_j - \frac{1}{2})$  is usually referred to as  $\xi$ -invariant and is often written as  $\xi_L$  in the theory of the Selberg class, but we do not use the letter  $\xi$  for the  $\xi$ -invariant of  $L \in \mathcal{S}$  to avoid confusion.



**2.2. Auxiliary functions for GRH.** Let  $z = u + iv$  be the complex variables relating with the variable  $s$  by  $s = 1/2 - iz$ . To work on  $\text{GRH}(L)$  for  $L \in \mathcal{S}_{\mathbb{R}}$ , we introduce the two parameter family of entire functions

$$E_L^{\omega, \nu}(z) := \xi_L(\tfrac{1}{2} + \omega - iz)^{\nu} = \xi_L(s + \omega)^{\nu} \quad (2.4)$$

parametrized by  $\omega \in \mathbb{R}_{>0}$  and  $\nu \in \mathbb{Z}_{>0}$ . Then,

$$(E_L^{\omega, \nu})^{\sharp}(z) = E_L^{\omega, \nu}(-z)$$

by the second equation in (2.3), and then  $(E_L^{\omega, \nu})^{\sharp}(z) = \epsilon_L^{\nu} \xi_L(\tfrac{1}{2} - \omega - iz)^{\nu}$  by the first equation in (2.3). Therefore, real entire functions  $A_L^{\omega, \nu}$  and  $B_L^{\omega, \nu}$  defined by (1.4) for  $E = E_L^{\omega, \nu}$  are even and odd, respectively. The following proposition is trivial from definition (2.4) and functional equations (2.3).

**Proposition 2.1.** *For  $L \in \mathcal{S}_{\mathbb{R}}$ ,  $\text{GRH}(L)$  holds if  $E_L^{\omega, 1} \in \overline{\mathbb{H}\mathbb{B}}$  for all  $\omega > 0$ .*

As easily found from (2.4) and definition of  $\overline{\mathbb{H}\mathbb{B}}$ , for fixed  $\omega > 0$ , if  $E_L^{\omega, \nu} \in \overline{\mathbb{H}\mathbb{B}}$  for some  $\nu \in \mathbb{Z}_{>0}$ , then  $E_L^{\omega, \nu} \in \overline{\mathbb{H}\mathbb{B}}$  for arbitrary  $\nu \in \mathbb{Z}_{>0}$ . Therefore,  $E_L^{\omega, 1}$  in Proposition 2.1 can be replaced by  $E_L^{\omega, \nu_{\omega}}$  defined for positive integers  $\nu_{\omega}$  indexed by  $\omega > 0$ . On the other hand,  $\overline{\mathbb{H}\mathbb{B}}$  in the statement can be replaced by  $\mathbb{H}\mathbb{B}$  without the changing of the meaning of the statement, although  $E_L^{\omega, 1} \in \overline{\mathbb{H}\mathbb{B}}$  is different from  $E_L^{\omega, 1} \in \mathbb{H}\mathbb{B}$  for individual  $\omega > 0$ .

As the converse of Proposition 2.1, we have the following.

**Proposition 2.2.** *Let  $L \in \mathcal{S}_{\mathbb{R}}$  and  $\nu \in \mathbb{Z}_{>0}$ . Then,  $E_L^{\omega, \nu}$  belongs to  $\mathbb{H}\mathbb{B}$  for every  $\omega > 1/2$  unconditionally and for every  $0 < \omega \leq 1/2$  under  $\text{GRH}(L)$ .*

*Proof.* First, we suppose that  $\omega > 1/2$ . Then  $E_L^{\omega, \nu}$  has no real zeros, since all zeros of  $\xi_L(s)$  lie in the vertical strip  $0 \leq \sigma \leq 1$ . On the other hand, we find that  $E_L^{\omega, \nu}$  satisfies (1.3) by applying [30, Theorem 4] to  $\xi_L(s)^{\nu}$ , since  $\xi_L(s)$  satisfies (2.3) and has only zeros in the strip  $0 \leq \sigma \leq 1$ . Hence  $E_L^{\omega, \nu} \in \mathbb{H}\mathbb{B}$ . The case of  $0 < \omega \leq 1/2$  is proved in a similar way under  $\text{GRH}(L)$ .  $\square$

The value  $\omega = 1/2$  in Proposition 2.2 comes from the trivial zero-free region of  $L \in \mathcal{S}$ . More precise relation between the zero-free region of  $L$  and the property  $E_L^{\omega, \nu} \in \mathbb{H}\mathbb{B}$  will be discussed in Proposition 4.3.

From Propositions 2.1 and 2.2, the validity of  $\text{GRH}(L)$  for  $L \in \mathcal{S}_{\mathbb{R}}$  is equivalent to the existence of a section  $\mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$ ;  $\omega \mapsto (\omega, \nu)$  such that  $E_L^{\omega, \nu} \in \overline{\mathbb{H}\mathbb{B}}$  for every  $\omega > 0$ . Hence  $\text{GRH}(L)$  will be established if the following three steps are completed for every point of a section:

- (3-1) Constructing the Hamiltonian  $H_L^{\omega, \nu}$  on  $I = [t_1, t_0]$  from  $E_L^{\omega, \nu}$  without  $\text{GRH}(L)$ ;
- (3-2) Constructing the solution  $(A_L^{\omega, \nu}(t, z), B_L^{\omega, \nu}(t, z))$  of the canonical system on  $I$  associated with  $H_L^{\omega, \nu}$  satisfying  $E_L^{\omega, \nu}(z) = A_L^{\omega, \nu}(t_1, z) - iB_L^{\omega, \nu}(t_1, z)$ ;
- (3-3) Showing that  $\lim_{t \rightarrow t_0} J_L^{\omega, \nu}(t; z, z) = 0$  for all  $z \in \mathbb{C}_+$ , where  $J_L^{\omega, \nu}(t; z, w)$  is the function defined by (1.6) for  $(A_L^{\omega, \nu}(t, z), B_L^{\omega, \nu}(t, z))$ .

The main results stated below concern each step in (3-1)~(3-3).

**2.3. Results on (3-1).** We define the function  $K_L^{\omega, \nu}$  on the real line by

$$\Theta_L^{\omega, \nu}(z) := \frac{(E_L^{\omega, \nu})^{\sharp}(z)}{E_L^{\omega, \nu}(z)} = \frac{E_L^{\omega, \nu}(-z)}{E_L^{\omega, \nu}(z)} = \epsilon_L^{\nu} \frac{\xi_L(\tfrac{1}{2} - \omega - iz)^{\nu}}{\xi_L(\tfrac{1}{2} + \omega - iz)^{\nu}} \quad (2.5)$$

and

$$K_L^{\omega, \nu}(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \Theta_L^{\omega, \nu}(u + iv) e^{-ix(u+iv)} du \quad (2.6)$$

for  $L \in \mathcal{S}_{\mathbb{R}}$ ,  $(\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$ . The integral on the right-hand side of (2.6) converges absolutely if  $v > 0$  is sufficiently large, and we have

$$K_L^{\omega, \nu}(x) = \epsilon_L^{\nu} \sum_{n=1}^{\lfloor \exp(x) \rfloor} \frac{q_L^{\omega, \nu}(n)}{\sqrt{n}} G_L^{\omega, \nu}(x - \log n) \quad (2.7)$$

for  $x > 0$  and  $K_L^{\omega, \nu}(x) = 0$  for  $x < 0$ , where  $\epsilon_L$  is the root number in (2.3),  $q_L^{\omega, \nu}(n)$  is an arithmetic function determined by the Dirichlet coefficients  $a_L(n)$  (non-archimedean information), and  $G_L^{\omega, \nu}$  is a certain explicit real-valued function having a support in  $[0, \infty)$  determined by the gamma factor  $\gamma_L$  (archimedean information).

We find that  $K_L^{\omega, \nu}$  is a continuous function if  $(\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$  satisfies

$$\nu \omega d_L > 1, \quad (2.8)$$

where  $d_L$  is the degree of  $L \in \mathcal{S}_{\mathbb{R}}$  (Proposition 4.1). This condition for  $(\omega, \nu)$  is technical but essential to the following construction of  $H_L^{\omega, \nu}$ . If  $(\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$  satisfies (2.8),

$$\mathcal{K}_L^{\omega, \nu}[t] : f(x) \mapsto \mathbf{1}_{(-\infty, t]}(x) \int_{-\infty}^t K_L^{\omega, \nu}(x + y) f(y) dy \quad (2.9)$$

defines a bounded operator on  $L^2(-\infty, t)$  for every  $t \in \mathbb{R}$ , where  $\mathbf{1}_A$  is the characteristic function of a set  $A$ . The study of  $\mathcal{K}_L^{\omega, \nu}[t]$  yields a canonical system as follows.

**Theorem 2.1.** *Let  $L \in \mathcal{S}_{\mathbb{R}}$  and  $(\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$  with (2.8). Then, for every  $t > 0$ , (2.9) defines a Hilbert-Schmidt type self-adjoint operator on  $L^2(-\infty, t)$  having a continuous kernel, and  $\mathcal{K}_L^{\omega, \nu}[t] = 0$  for  $t \leq 0$ . Moreover, there exists  $\tau = \tau(L; \omega, \nu) > 0$  such that both  $\pm 1$  are not the eigenvalues of  $\mathcal{K}_L^{\omega, \nu}[t]$  for every  $t \in [0, \tau)$ . In particular, the Fredholm determinant  $\det(1 \pm \mathcal{K}_L^{\omega, \nu}[t])$  does not vanish for every  $t \in [0, \tau)$ .*

The first half will be proved by applying the argument in Section 3.1 to  $K_L^{\omega, \nu}$  under Proposition 4.1. The latter half is Proposition 4.2. Propositions 4.1 and 4.2 are proved in Section 4. By (2.9), to understand the operator  $\mathcal{K}_L^{\omega, \nu}[t]$ , we need only the values of  $K_L^{\omega, \nu}$  on  $[0, 2t)$  that are determined by the information on the gamma factor  $\gamma_L$  and finitely many coefficient  $q_L^{\omega, \nu}(n)$ 's by (2.7).

By Theorem 2.1, the Hamiltonian  $H_L^{\omega, \nu}$  on  $[0, \tau)$  is defined by

$$H_L^{\omega, \nu}(t) := \begin{bmatrix} 1/\gamma_L^{\omega, \nu}(t) & 0 \\ 0 & \gamma_L^{\omega, \nu}(t) \end{bmatrix}, \quad \gamma_L^{\omega, \nu}(t) := \left( \frac{\det(1 + \mathcal{K}_L^{\omega, \nu}[t])}{\det(1 - \mathcal{K}_L^{\omega, \nu}[t])} \right)^2, \quad (2.10)$$

and it defines the canonical system

$$-\frac{d}{dt} \begin{bmatrix} A(t, z) \\ B(t, z) \end{bmatrix} = z \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} H_L^{\omega, \nu}(t) \begin{bmatrix} A(t, z) \\ B(t, z) \end{bmatrix}, \quad z \in \mathbb{C} \quad (2.11)$$

on  $[0, \tau)$ . By definition, the Hamiltonian  $H_L^{\omega, \nu}$  has no  $H_L^{\omega, \nu}$ -indivisible intervals, that is, all points of  $[0, \tau)$  are regular. Next, we construct the unique solution of the canonical system recovering the entire function  $E_L^{\omega, \nu}$ .

**2.4. Results on (3-2).** Let  $\tau$  be the number in Theorem 2.1. Then, two integral equations

$$(1 \pm \mathcal{K}_L^{\omega, \nu}[t])\varphi_t^{\pm}(x) = \mathbf{1}_{(-\infty, t]}(x) K_L^{\omega, \nu}(x + t)$$

for unknown functions  $\varphi_t^{\pm} \in L^2(-\infty, t)$  have unique solutions for every  $t \in [0, \tau)$ , since  $1 \pm \mathcal{K}_L^{\omega, \nu}[t]$  are invertible. We extend the solutions  $\varphi_t^{\pm}$  to continuous functions on  $\mathbb{R}$  by

$$\phi^{\pm}(t, x) = K_L^{\omega, \nu}(x + t) \mp \int_{-\infty}^t K_L^{\omega, \nu}(x + y) \varphi_t^{\pm}(y) dy.$$



Then, we find that  $|\phi^\pm(t, x)| \ll e^{c|x|}$  for some  $c > 0$  which may depend on  $t$  (Lemma 3.7 and Proposition 4.1). Therefore, two functions defined by

$$\begin{aligned} A_L^{\omega, \nu}(t, z) &:= \frac{1}{2} \gamma_L^{\omega, \nu}(t)^{1/2} E_L^{\omega, \nu}(z) \left( e^{izt} + \int_t^\infty \phi^+(t, x) e^{izx} dx \right), \\ -iB_L^{\omega, \nu}(t, z) &:= \frac{1}{2} \gamma_L^{\omega, \nu}(t)^{-1/2} E_L^{\omega, \nu}(z) \left( e^{izt} - \int_t^\infty \phi^-(t, x) e^{izx} dx \right) \end{aligned} \quad (2.12)$$

are analytic functions in the upper half-plane  $\Im(z) > c'$  for  $t \in [0, \tau)$ .

**Theorem 2.2.** *Let  $L \in \mathcal{S}_\mathbb{R}$  and  $(\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$  with (2.8). We define  $E_L^{\omega, \nu}$  by (2.4), and then define  $A_L^{\omega, \nu}$  and  $B_L^{\omega, \nu}$  by (1.4) for  $E = E_L^{\omega, \nu}$ . Let  $\tau = \tau(L; \omega, \nu)$  be the positive real number in Theorem 2.1. Let  $A_L^{\omega, \nu}(t, z)$  and  $B_L^{\omega, \nu}(t, z)$  be the families of functions defined by (2.12). Then,*

- (1)  $A_L^{\omega, \nu}(t, z)$  and  $B_L^{\omega, \nu}(t, z)$  are extended to real entire functions as a function of  $z$  for every  $t \in [0, \tau)$ ,
- (2)  $A_L^{\omega, \nu}(t, z)$  is even and  $B_L^{\omega, \nu}(t, z)$  is odd as a function of  $z$  for every  $t \in [0, \tau)$ ,
- (3)  $A_L^{\omega, \nu}(t, z)$  and  $B_L^{\omega, \nu}(t, z)$  are continuous and piecewise continuously differentiable functions as functions of  $t$  for every  $z \in \mathbb{C}$ ,
- (4)  ${}^t(A_L^{\omega, \nu}(t, z), B_L^{\omega, \nu}(t, z))$  solve the canonical system (2.11) on  $[0, \tau)$ ,
- (5)  $A_L^{\omega, \nu}(0, z) = A_L^{\omega, \nu}(z)$ ,  $B_L^{\omega, \nu}(0, z) = B_L^{\omega, \nu}(z)$ , and thus

$$E_L^{\omega, \nu}(z) = A_L^{\omega, \nu}(0, z) - iB_L^{\omega, \nu}(0, z). \quad (2.13)$$

Theorem 2.2 is a generalization of [40, Theorem 2.3] which only deal with the case of  $L = \zeta$  and  $\nu = 1$ . The condition (2.8) is  $\omega > 1$  for  $L = \zeta$  and  $\nu = 1$  by  $d_\zeta = 1$ .

Theorem 2.2 will be proved by applying Theorem 3.1 to the entire function  $E_L^{\omega, \nu}$  under Propositions 4.1 and 4.2 that are proved in Section 4.

**2.5. Results on (3-3).** By the above results, we obtain the Hamiltonian  $H_L^{\omega, \nu}$  defined on  $[0, \tau)$  and the solution  ${}^t(A_L^{\omega, \nu}(t, z), B_L^{\omega, \nu}(t, z))$  of the canonical system associated with  $H_L^{\omega, \nu}$  satisfying (2.13) without GRH( $L$ ) for  $L \in \mathcal{S}_\mathbb{R}$ . However, it is not enough to conclude that  $E_L^{\omega, \nu}(z) \in \overline{\mathbb{H}\mathbb{B}}$  for small  $\omega > 0$  because we have no information about the solution  ${}^t(A_L^{\omega, \nu}(t, z), B_L^{\omega, \nu}(t, z))$  at the right endpoint  $t = \tau$ . However, we can not expect a nice result at  $t = \tau$ , because it seems that the interval  $[0, \tau)$  in Theorem 2.1 is not the one in  $\text{Inv}(E_L^{\omega, \nu})$ . In fact, the result [40, Theorem 2.3] suggests that the genuine interval in  $\text{Inv}(E_L^{\omega, \nu})$  is the half-line  $[0, \infty)$ .

If we suppose that  $E_L^{\omega, \nu} \in \mathbb{H}\mathbb{B}$ , the Fredholm determinant  $\det(1 \pm K_L^{\omega, \nu}[t])$  does not vanish for every  $t \geq 0$  (Proposition 4.4. See also Proposition 2.2 and Section 9.3). Thus the Hamiltonian  $H_L^{\omega, \nu}$  of (2.10) extends to a Hamiltonian on  $I = [0, \infty)$  having no  $H_L^{\omega, \nu}$ -indivisible intervals and the solution  ${}^t(A_L^{\omega, \nu}(t, z), B_L^{\omega, \nu}(t, z))$  of (2.12) also extends to the solution of the canonical system (2.11) on  $I$ . Moreover we have the following result which will be proved in Section 6.

**Theorem 2.3.** *Let  $L \in \mathcal{S}_\mathbb{R}$  and  $(\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$  with (2.8). Assume that  $E_L^{\omega, \nu} \in \mathbb{H}\mathbb{B}$  and define  $J_L^{\omega, \nu}(t; z, w)$  by (1.6) for  $(A_L^{\omega, \nu}(t, z), B_L^{\omega, \nu}(t, z))$ . Then  $J_L^{\omega, \nu}(t; z, w) \neq 0$  for any  $t \geq 0$  and  $\lim_{t \rightarrow \infty} J_L^{\omega, \nu}(t; z, w) = 0$  for every fixed  $z, w \in \mathbb{C}$ .*

**2.6. Equivalent condition for GRH.** Considering Theorem 2.2, Theorem 2.3 and Proposition 3.4 below,  $E_L^{\omega, \nu} \in \overline{\mathbb{H}\mathbb{B}}$  is equivalent to the condition that  $H_L^{\omega, \nu}$  is extended to a Hamiltonian on  $[0, \infty)$  and  $\lim_{t \rightarrow \infty} J_L^{\omega, \nu}(t; z, w) = 0$  if  $\nu \omega d_L > 1$ . Hence we obtain the following equivalent condition for GRH( $L$ ) for  $L \in \mathcal{S}_\mathbb{R}$  by noting that the range of  $\omega$  in Proposition 2.1 can be relaxed to a decreasing sequence tending to 0.

**Theorem 2.4.** *The validity of GRH( $L$ ) for  $L \in \mathcal{S}_\mathbb{R}$  is equivalent to the condition that there exists a sequence  $(\omega_n, \nu_n) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$ ,  $n \geq 1$ , such that*

- (1)  $\omega_m < \omega_n$  if  $m > n$  and  $\omega_n \rightarrow 0$  as  $n \rightarrow \infty$ ,

- (2)  $\nu_n \omega_n d_L > 1$ ,
- (3)  $\det(1 \pm \mathbf{K}_L^{\omega_n, \nu_n}[t]) \neq 0$  for every  $t \geq 0$  (thus  $H_L^{\omega_n, \nu_n}$  extends to a Hamiltonian on  $[0, \infty)$  having no  $H_L^{\omega_n, \nu_n}$ -indivisible intervals), and
- (4)  $\lim_{t \rightarrow \infty} J_L^{\omega_n, \nu_n}(t; z, z) = 0$  for every  $z \in \mathbb{C}_+$ .

The detailed proof of Theorem 2.4 will be given in Section 7. Also, a variant of Theorem 2.4 is stated in Section 9.1.

### 3. INVERSE PROBLEM FOR SOME SPECIAL CANONICAL SYSTEM

Let  $A$  and  $B$  be entire functions defined by (1.4) for an entire function  $E$ . The goal of this section is the construction of a triple  $\text{Inv}(E)^b$  consisting of some possibly infinite interval  $I = [0, \tau)$  ( $0 < \tau \leq \infty$ ),  $\text{Sym}_2^+(\mathbb{R})$ -valued function  $H$  defined on  $I$  and the unique solution  ${}^t(A(t, z), B(t, z))$  of the canonical system (1.5) on  $I$  associated with  $H$  satisfying  $(A(0, z), B(0, z)) = (A(z), B(z))$ , where  $\text{Sym}_2^+(\mathbb{R})$  is the set of all positive definite matrices in  $\text{Sym}_2(\mathbb{R})$ . The way of construction is similar to [13] and [40].

The most important point is that we will impose several conditions for  $E$  but  $E \in \overline{\text{HIB}}$  is not necessary to the following construction of  $\text{Inv}(E)^b$  differ from de Branges' theory for the inverse spectral problem  $\text{Inv}(E)$ .

**3.1. Basic assumptions.** Let  $L^p(I)$  be the  $L^p$ -space on an interval  $I$  with respect to the Lebesgue measure. If  $J \subset I$ , we regard  $L^p(J)$  as a subspace of  $L^p(I)$  by the extension by zero. We denote by

$$(\mathbf{F}f)(z) = \int_{-\infty}^{\infty} f(x) e^{ixz} dx, \quad (\mathbf{F}^{-1}g)(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u) e^{-ixu} du$$

the Fourier integral and inverse Fourier integral, respectively. We use the same notation for the Fourier transforms on  $L^1(\mathbb{R})$  and  $L^2(\mathbb{R})$  if no confusion arises. If we understand the right-hand sides in  $L^2$ -sense, they provide isometries on  $L^2(\mathbb{R})$  up to a constant multiple:  $\|\mathbf{F}f\|_{L^2(\mathbb{R})}^2 = 2\pi\|f\|_{L^2(\mathbb{R})}^2$ ,  $\|\mathbf{F}^{-1}f\|_{L^2(\mathbb{R})}^2 = (2\pi)^{-1}\|f\|_{L^2(\mathbb{R})}^2$ .

We impose several conditions for  $E$  throughout this section, because the existence of a Hamiltonian is not guaranteed for general entire function  $E$ .

The first is the following condition:

- (K1) There exists a real-valued continuously differentiable function  $\varrho$  on  $\mathbb{R}$  such that  $|\varrho(x)| \ll e^{-n|x|}$  for any  $n > 0$  and  $E(z) = (\mathbf{F}\varrho)(z)$  for all  $z \in \mathbb{C}$ .

Here “ $\ll$ ” stands for the Vinogradov symbol which will be used often as well as the Landou symbols “ $O$ ” and “ $o$ ”. We have  $E^\sharp(z) = E(-z)$  from (K1). Thus,  $A(z)$  is even and  $B(z)$  is odd under (K1). We define

$$\Theta(z) = \Theta_E(z) := \frac{E^\sharp(z)}{E(z)}.$$

Then,  $\Theta(0) = 1$ ,  $\Theta(z)\Theta(-z) = 1$  for  $z \in \mathbb{C}$  and

$$|\Theta(u)| = 1 \quad \text{for } u \in \mathbb{R} \tag{3.1}$$

by definition and (K1). In addition we suppose the following conditions:

- (K2) There exists a real-valued continuous function  $K$  defined on the real line such that  $|K(x)| \ll \exp(c|x|)$  for some  $c \geq 0$  and  $\Theta(z) = (\mathbf{F}K)(z)$  holds for  $\Im(z) > c$ .
- (K3)  $K$  vanishes on  $(-\infty, 0)$ ,
- (K4)  $K$  is continuously differentiable outside a discrete subset  $\Lambda \subset \mathbb{R}$  and  $|K'|$  is locally integrable on  $\mathbb{R}$ .

Under the above conditions, the map

$$\mathbf{K}[t] : f(x) \mapsto 1_{(-\infty, t]}(x) \int_{-\infty}^t K(x+y) f(y) dy$$

defines a Hilbert–Schmidt operator on  $L^2(-\infty, t)$  for every  $t > 0$ . In fact, the Hilbert–Schmidt norm of  $K[t]$  is finite:

$$\int_{-\infty}^t \int_{-\infty}^t |K(x+y)|^2 dx dy \leq \int_{-t}^t dy \int_{-2t}^{2t} |K(x)|^2 dx = 2t \int_0^{2t} |K(x)|^2 dx < \infty$$

by the continuity of  $K$  in (K2) and (K3). Because the kernel  $K(x+y)$  is real-valued and symmetric,  $K[t]$  is self-adjoint. For  $t \leq 0$ , we understand  $K[t] = 0$  by (K3).

Finally, we suppose the following condition in addition to (K1)~(K4):

(K5) There exists  $0 < \tau \leq \infty$  such that both  $\pm 1$  are not eigenvalues of  $K[t]$  for every  $t < \tau$ .

The requirement for eigenvalues of  $K[t]$  in (K5) is trivial for  $t \leq 0$  by  $K[t] = 0$ . The set of entire functions satisfying (K1)~(K5) is not empty. In fact, it will be shown in Section 4 that  $E_L^{\omega, \nu}$  of (2.4) satisfies these five conditions for every  $L \in \mathcal{S}_{\mathbb{R}} \neq \emptyset$  under (2.8).

Here we should mention that (K5) plays the roll of the condition  $E \in \overline{\mathbb{H}\mathbb{B}}$  in the following construction of  $\text{Inv}(E)^b$ , but (K5) does not imply  $E \in \overline{\mathbb{H}\mathbb{B}}$  in general, because  $\tau$  in (K5) may not be the right endpoint of the interval in  $\text{Inv}(E)$  and hence a nice information at the point  $\tau$  can not be expected.

Under the above five conditions (K1)~(K5), we will construct a triple  $\text{Inv}(E)^b$  by using solutions of some family of integral equations after the next subsection.

**3.2. Meromorphic inner functions.** Let  $H^\infty = H^\infty(\mathbb{C}_+)$  be the space of all bounded analytic functions in  $\mathbb{C}_+$ . A function  $\Theta \in H^\infty$  is called an *inner function* in  $\mathbb{C}_+$  if  $\lim_{y \rightarrow 0+} |\Theta(x+iy)| = 1$  for almost all  $x \in \mathbb{R}$  with respect to the Lebesgue measure. If an inner function  $\Theta$  in  $\mathbb{C}_+$  is extended to a meromorphic function in  $\mathbb{C}$ , it is called a *meromorphic inner function* in  $\mathbb{C}_+$ .

**Lemma 3.1.** *Let  $E$  be an entire function. The meromorphic function  $\Theta = E^\sharp/E$  is a meromorphic inner function in  $\mathbb{C}_+$  if and only if  $E$  belongs to  $\overline{\mathbb{H}\mathbb{B}}$ .*

*Proof.* If  $E \in \overline{\mathbb{H}\mathbb{B}}$ ,  $\Theta = E^\sharp/E$  is a meromorphic inner function by (1.3) and (3.1). Conversely, if  $\Theta$  is a meromorphic inner function, there exists  $E \in \mathbb{H}\mathbb{B}$  such that  $\Theta = E^\sharp/E$  ([17, §2.3 and §2.4]). Here  $E$  is not unique and we may choose  $E \in \mathbb{H}\mathbb{B}$ , because we can factorize  $E \in \overline{\mathbb{H}\mathbb{B}}$  as  $E = E_0 E_1$  such that  $E_0 \in \mathbb{H}\mathbb{B}$ ,  $E_1$  having only real zeros and  $E_1^\sharp = E_1$  by using usual factorization theorem for functions in  $H^\infty$ .  $\square$

**Lemma 3.2.** *Let  $E$  be an entire function satisfying (K1)~(K3). Suppose that  $\Theta = E^\sharp/E$  is inner in  $\mathbb{C}_+$ . Define  $Kf$  by the integral*

$$(Kf)(x) = \int_{-\infty}^{\infty} K(x+y) f(y) dy \quad (3.2)$$

for  $f$  in the space  $C_c^\infty(\mathbb{R})$  of all compactly supported smooth function on  $\mathbb{R}$ . Then  $Kf$  belongs to  $L^2(\mathbb{R})$ , and the linear map  $f \mapsto Kf$  is extended to the isometry  $K : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  satisfying  $K^2 = \text{id}$  and

$$(FKf)(z) = \Theta(z) (Ff)(-z) \quad (3.3)$$

for  $z \in \mathbb{R}$ . Moreover, (3.3) holds for  $\Im(z) \geq 0$ , if  $f \in L^2(\mathbb{R})$  has a support in  $(-\infty, t]$  for some  $t \in \mathbb{R}$ .

*Proof.* If  $f \in C_c^\infty(\mathbb{R})$ , we have

$$\begin{aligned} (FKf)(z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x+y) e^{izx} dx f(y) dy \\ &= \int_{-\infty}^{\infty} K(x) e^{izx} dx \int_{-\infty}^{\infty} f(y) e^{-izy} dy = \Theta(z) F(-z) \end{aligned}$$

for  $\Im(z) > c$  by (K2), where  $F = Ff$ . In the right-hand side,  $F(-z)$  is an entire function satisfying  $F(-z) = O(|z|^{-n})$  as  $|z| \rightarrow \infty$  in any horizontal strip  $c_1 \leq \Im(z) \leq c_2$  for arbitrary fixed  $n > 0$ . Therefore, we find that  $Kf$  belongs to  $L^2(\mathbb{R})$  by applying the Fourier inversion formula to  $\Theta(z)F(-z)$  along a line  $\Im(z) = c' > c$  and then moving the path of integration to the real line  $\Im(z) = 0$ , since  $\Theta$  is inner in  $\mathbb{C}_+$ . Moreover

$$\|Kf\|^2 = \frac{1}{2\pi} \|\Theta(\cdot)F(-\cdot)\|^2 = \frac{1}{2\pi} \|F\|^2 = \|f\|^2$$

by (3.1), where  $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R})}$ . Thus  $f \mapsto Kf$  is extended to  $f \in L^2(\mathbb{R})$  by the continuity and the denseness of  $C_c^\infty(\mathbb{R})$  in  $L^2(\mathbb{R})$ . The extended operator is obviously isometric. Equality (3.3) holds for real  $z$  by the continuity, and it implies  $K^2 = \text{id}$ .

Suppose that  $f \in L^2(\mathbb{R})$  has a support in  $(-\infty, t]$  for some  $t \in \mathbb{R}$ . Then  $Kf$  belongs to  $L^2(\mathbb{R})$  and has a support in  $[-t, \infty)$  by (K3). Therefore the left-hand side of (3.3) is defined by the Fourier integral and analytic in  $\mathbb{C}_+$ . On the other hand,  $F(-z)$  in the right-hand side of (3.3) is also defined by the Fourier integral and analytic in  $\mathbb{C}_+$ . Hence both sides of (3.3) are analytic functions in  $\mathbb{C}_+$ , and they are equal on the real line. Thus equality (3.3) holds for  $\Im(z) \geq 0$ .  $\square$

**3.3. Existence and uniqueness of solutions of integral equations.** We suppose that  $E$  satisfies (K1)~(K5) throughout this and the later subsections. In particular, we understand that  $\tau$  is the number in (K5).

**Lemma 3.3.** *Let  $\varepsilon \in \{\pm 1\}$  and  $t \in (0, \tau)$ . Then the integral equation*

$$X(x) + \varepsilon \mathbf{1}_{(-\infty, t]}(x) \int_{-\infty}^t K(x+y)X(y) dy = \mathbf{1}_{(-\infty, t]}(x)K(x+t) \quad (3.4)$$

*has the unique solution  $X = \varphi_t^\varepsilon$  in  $L^2(-\infty, t)$ . The solution  $\varphi_t^\varepsilon$  is a real-valued continuous function on  $(-\infty, t]$  vanishing on  $(-\infty, -t)$ .*

*Proof.* By the assumption for  $E$ ,  $K[t]$  is a compact operator on  $L^2(-\infty, t)$  such that both  $\pm 1$  belong to its resolvent set. Therefore, integral equation (3.4) has the unique solution  $\varphi_t^\varepsilon$  in  $L^2(-\infty, t)$  by the Fredholm alternative. We easily obtain the other properties of the solution  $\varphi_t^\varepsilon$  from (3.4), (K2) and (K3).  $\square$

**Lemma 3.4.** *Let  $\varepsilon \in \{\pm 1\}$  and  $t \in (0, \tau)$ . Then, for arbitrary  $t < s < \tau$ , the equation*

$$X(x) + \varepsilon \mathbf{1}_{(-\infty, s]}(x) \int_{-\infty}^t K(x+y)X(y) dy = \mathbf{1}_{(-\infty, s]}(x)K(x+t) \quad (3.5)$$

*has the unique solution  $X = \tilde{\varphi}_t^\varepsilon$  in  $L^2(-\infty, s)$ . The solution  $\tilde{\varphi}_t^\varepsilon$  is a real-valued continuous function on  $(-\infty, s]$  satisfying  $\tilde{\varphi}_t^\varepsilon = \varphi_t^\varepsilon$  on  $(-\infty, t]$  for the unique solution  $\varphi_t^\varepsilon$  in Lemma 3.3.*

*Proof.* The solution  $\varphi_t^\varepsilon$  of Lemma 3.3 is extended to the continuous solution  $\tilde{\varphi}_t^\varepsilon$  of (3.5) on  $(-\infty, s]$  by

$$\tilde{\varphi}_t^\varepsilon(x) = \mathbf{1}_{(-\infty, s]}(x)K(x+t) - \varepsilon \mathbf{1}_{(-\infty, s]}(x) \int_{-\infty}^t K(x+y)\varphi_t^\varepsilon(y) dy. \quad (3.6)$$

The right-hand side belongs to  $L^2(-\infty, s)$  by the Cauchy-Schwartz inequality, since  $K$  belongs to  $L^2(-\infty, s')$  for every  $s' \in \mathbb{R}$  and the integral on the right-hand side vanishes for almost every  $x < -t$  by (K2) and (K3). Clearly,  $\tilde{\varphi}_t^\varepsilon = \varphi_t^\varepsilon$  on  $(-\infty, t]$ . Conversely, equality (3.6) shows that every solution of (3.4) on  $L^2(-\infty, s)$  is determined by its restriction on  $(-\infty, t)$ . Hence the uniqueness of solutions follows from Lemma 3.3. By the way of the extension,  $\tilde{\varphi}_t^\varepsilon$  is real-valued.  $\square$

In what follows, we denote by  $\phi^\varepsilon(t, x)$  the extension of  $\varphi_t^\varepsilon(x)$  to  $\mathbb{R}$  for  $t \in (0, \tau)$  if no confusion arises. That is,  $\phi^\varepsilon(t, x)$  is given by

$$\phi^\varepsilon(t, x) = K(x+t) - \varepsilon \int_{-\infty}^t K(x+y) \varphi_t^\varepsilon(y) dy. \quad (3.7)$$

The extended solution  $\phi^\varepsilon(t, x)$  are real-valued, since  $K(x)$  and  $\varphi_t^\varepsilon(x)$  are real-valued. We take the convention that

$$\phi^\varepsilon(0, x) = K(x), \quad \varepsilon \in \{\pm 1\}.$$

This convention is compatible with Lemma 3.3 and 3.4, since integral equation (3.4) for  $t = 0$  should be  $X(x) = \mathbf{1}_{(-\infty, t]}(x)K(x)$  by (K3). Therefore,  $\varphi_0^\varepsilon = \mathbf{1}_{(-\infty, t]} \cdot K$  and the extension  $\tilde{\varphi}_0^\varepsilon$  to  $\mathbb{R}$  should be  $K$  by (3.6).

At this point, it is not clear whether  $\phi^\varepsilon(t, x)$  belongs to  $L^2(\mathbb{R})$  as a function of  $x$ , but at least, the restriction of  $\phi^\varepsilon(t, x)$  to  $(-\infty, t]$  satisfy (3.4) and  $\phi^\varepsilon(t, x)$  is the unique continuous solution of

$$\phi^\varepsilon(t, x) + \varepsilon \int_{-\infty}^t K(x+y) \phi^\varepsilon(t, y) dy = K(x+t). \quad (3.8)$$

We note that (3.8) and the integral equation

$$\int_{-\infty}^{\infty} \varrho(x-y) \phi^\varepsilon(t, y) dy + \varepsilon \int_{-\infty}^t \varrho(-x-y) \phi^\varepsilon(t, w) dy = \varrho(-x-t) \quad (3.9)$$

for  $\varrho$  in (K1) have the same solution  $\phi^\varepsilon(t, x)$ , since the Fourier transforms of (3.8) and (3.9) give the same equation

$$E(z) \cdot \int_{-\infty}^{\infty} \phi^\varepsilon(t, x) e^{izx} dx + \varepsilon \cdot E^\sharp(z) \cdot \int_{-\infty}^t \phi^\varepsilon(t, x) e^{-izx} dx = e^{-itz} E^\sharp(z) \quad (3.10)$$

if we note that  $\int_{-\infty}^{\infty} \phi^\pm(t, x) e^{izx} dx$  is defined for  $\Im(z) \gg 0$  (Lemma 3.7 below) and  $\int_{-\infty}^t \phi^\pm(t, x) e^{izx} dx = \int_{-t}^t \phi^\pm(t, x) e^{izx} dx$  is defined for every  $z \in \mathbb{C}$ .

**3.4. Differentiability of solutions.** We handle the differentiability of the extended solution  $\phi^\varepsilon(t, x)$  for both variables under (K1)~(K5).

**Lemma 3.5.** *The extended solution  $\phi^\varepsilon(t, x)$  is continuous on  $\mathbb{R}$  and continuously differentiable on  $\mathbb{R} \setminus \{\lambda - t \mid \lambda \in \Lambda\}$  as a function of  $x$ , where  $\Lambda$  is the set in (K4).*

*Proof.* The continuity of  $\phi^\varepsilon(t, x)$  in  $x$  is obvious from the definition (3.7) because of the continuity of  $K$  and  $\varphi_t^\varepsilon$ . On the other hand,

$$\frac{\partial}{\partial x} \phi^\varepsilon(t, x) = K'(x+t) - \varepsilon \int_{-\infty}^t K'(x+y) \varphi_t^\varepsilon(y) dy$$

by (3.7). The integral on the right hand side defines a continuous function of  $x$ , since  $|K'(x+y)|$  is integrable on  $(-\infty, t]$  by (K3) and (K4), and  $\varphi_t^\varepsilon$  is a continuous function vanishing on  $(-\infty, -t)$ . Hence  $\frac{\partial}{\partial x} \phi^\varepsilon(t, x)$  is continuous on  $\mathbb{R}$  as a function of  $x$  except for points in  $\{\lambda - t \mid \lambda \in \Lambda\}$ .  $\square$

We investigate the differentiability of  $\phi^\varepsilon(t, x)$  with respect to  $t$  by using the kernel  $R(x, y; \mu; t)$  of the resolvent  $(1 - \mu K[t])^{-1}$ , because the solution  $\varphi_t^\varepsilon$  of (3.4) is related to  $R(x, y; \mu; t)$  as follows.

Let  $\Omega_t = (-\infty, t] \times (-\infty, t]$ . We introduce the notation

$$K \begin{pmatrix} x_1, x_2, \dots, x_n \\ y_1, y_2, \dots, y_n \end{pmatrix} = \det \begin{pmatrix} K(x_1, y_1) & K(x_1, y_2) & \cdots & K(x_1, y_n) \\ K(x_2, y_1) & K(x_2, y_2) & \cdots & K(x_2, y_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n, y_1) & K(x_n, y_2) & \cdots & K(x_n, y_n) \end{pmatrix}$$

as usual for the kernel  $K(x, y) = K(x + y)$ . The Fredholm determinant  $d(\mu; t)$  and the first Fredholm minor  $D(x, y; \mu; t)$  of the continuous kernel  $K(x, y)$  on  $\Omega_t$  are defined by

$$d(\mu; t) = \sum_{n=0}^{\infty} d_n(t) \mu^n, \quad (3.11)$$

$$D(x, y; \mu; t) = \sum_{n=0}^{\infty} D_n(x, y; t) \mu^n, \quad (3.12)$$

where  $d_0(t) = 1$ ,  $D_0(x, y; t) = K(x, y)$  and

$$d_n(t) = \frac{(-1)^n}{n!} \int_{-\infty}^t \cdots \int_{-\infty}^t K \begin{pmatrix} x_1, x_2, \dots, x_n \\ x_1, x_2, \dots, x_n \end{pmatrix} dx_1 \dots dx_n \quad (n \geq 1), \quad (3.13)$$

$$D_n(x, y; t) = \frac{(-1)^n}{n!} \int_{-\infty}^t \cdots \int_{-\infty}^t K \begin{pmatrix} x, x_1, \dots, x_n \\ y, x_1, \dots, x_n \end{pmatrix} dx_1 \dots dx_n \quad (n \geq 1). \quad (3.14)$$

Integrals on the right hand sides are converges absolutely by (K2) and (K3). In particular, the kernel  $D_n(x, y; t)$  is continuous in  $(x, y)$ . Note that  $d_n(t) = D_n(x, y; t) = 0$  for every  $n \geq 1$  if  $t \leq 0$  by (K3). It is well-known that the series (3.11) and (3.12) converge uniformly and absolutely in  $\mu$  and  $(x, y, \mu)$  respectively, when  $\mu$  is confined in a compact subset of  $\mathbb{C}$  (see [37, Theorem 5.3.1], for example). A standard way of the proof of such facts also provides the continuity for  $t$  as follows. Put

$$M(t) = t \cdot \sup_{(x, y) \in \Omega_t} |K(x, y)|.$$

This is defined well by (K2) and (K3), and defines a continuous function of  $t$ . We have

$$|d_n(t)| \leq \frac{n^{\frac{1}{2}n} M(t)^n}{n!}, \quad |D_n(x, y; t)| \leq \frac{n^{\frac{1}{2}n} M(t)^n}{n!} \frac{M(t)}{t}$$

for  $(x, y) \in \Omega_t$  and  $t \in [0, \tau)$  by (K3), definitions (3.13), (3.14) and Hadamard's inequality ([37, Theorem 5.2.1]) which assert that  $|\det(a_{ij})_{1 \leq i, j \leq n}|^2 \leq n^n M^{2n}$  if  $|a_{ij}| \leq M$  ( $1 \leq i, j \leq n$ ). Therefore the series of (3.11) (resp. (3.12)) converges absolutely and uniformly on a compact subset of  $(\mu, t) \in \mathbb{C} \times [0, \tau)$  (resp.  $((x, y), t, \mu) \in \Omega_t \times [0, \tau) \times \mathbb{C}$ ). In particular,  $D(x, y; \mu; t)$  and  $d(\mu; t)$  are continuous in all variables.

If  $d(\mu; t) \neq 0$ , that is  $1/\mu$  is not an eigenvalue of  $K[t]$ , the resolvent kernel  $R(x, y; \mu; t)$  is defined by

$$R(x, y; \lambda; t) = \frac{D(x, y; \lambda; t)}{d(\lambda; t)}. \quad (3.15)$$

Note that  $d(\pm 1; t) \neq 0$  for every  $t \in [0, \tau)$  by the assumption (K5). By the general theory of integral equations,  $R(x, y; \mu; t)$  satisfies integral equations

$$\begin{aligned} R(x, y; \mu; t) - \mu \int_{-\infty}^t K(x + z) R(z, y; \mu; t) dz &= K(x + y), \\ R(x, y; \mu; t) - \mu \int_{-\infty}^t K(z + y) R(x, z; \mu; t) dz &= K(x + y) \end{aligned} \quad (3.16)$$

for  $((x, y), \mu) \in \Omega_t \times \mathbb{C}$  (see Smithies [37, Chap. V], Lax [31, Chap. 24], for example). By taking  $y = t$  and  $\mu = -\varepsilon$  in the first equation of (3.16), we have

$$R(x, t; -\varepsilon; t) + \varepsilon \int_{-\infty}^t K(x + z) R(z, t; -\varepsilon; t) dz = K(x + t).$$

Therefore, we obtain

$$\varphi_t^\varepsilon(x) = R(x, t; -\varepsilon; t) \quad (3.17)$$



for  $x \in (-\infty, t]$  by the uniqueness of solutions of (3.4). In particular, we obtain the continuity of  $\varphi_t^\varepsilon(x)$  for  $x$  again and the continuity of  $\varphi_t^\varepsilon(x)$  for  $t$ . We have

$$\lim_{t \rightarrow 0^+} \varphi_t^\varepsilon(t) = \lim_{t \rightarrow 0^+} R(t, t; -\varepsilon; t) = 0, \quad (3.18)$$

since  $R(x, y; \mu; t) = K(x + y) + o(1)$  as  $t \rightarrow 0^+$  by (K3).

**Lemma 3.6.** *The extended solution  $\phi^\varepsilon(t, x)$  is continuous on  $[0, \tau)$  and continuously differentiable on  $(0, \tau)$  except for points in  $\{\lambda - x \mid \lambda \in \Lambda\}$  as a function of  $t$ .*

*Proof.* The first half of the lemma follows from (3.7) and the above argument. If  $d(\mu; t) \neq 0$  for every  $t \in [0, \tau)$  as well as  $\mu = \pm 1$ , we have

$$\frac{\partial}{\partial t} R(x, t; \mu; t) = \frac{\frac{\partial}{\partial t} D(x, t; \mu; t) d(\mu; t) - D(x, t; \mu; t) \frac{\partial}{\partial t} d(\mu; t)}{d(\mu; t)^2}$$

by (3.15). Therefore, in order to prove the latter half of the lemma, it is sufficient to prove (i) the existence and the continuity of  $\frac{\partial}{\partial t} d(\lambda; t)$  and (ii) the existence, the continuity and the integrability of  $\frac{\partial}{\partial t} D(x, t; \lambda; t)$  in  $x$  by (3.7), (3.17) and

$$\frac{\partial}{\partial t} \int_{-\infty}^t K(x + y) \varphi_t^\varepsilon(y) dy = K(x + t) \varphi_t^\varepsilon(t) + \int_{-\infty}^t K(x + y) \frac{\partial}{\partial t} \varphi_t^\varepsilon(y) dy.$$

We prove (i). By definition (3.11) and (3.13), we have

$$\begin{aligned} \frac{\partial}{\partial t} d(\mu; t) &= -\mu K(t, t) + \sum_{n=2}^{\infty} \frac{(-\mu)^n}{n!} \left\{ \int_{-\infty}^t \cdots \int_{-\infty}^t K \begin{pmatrix} t, x_2, \dots, x_n \\ t, x_2, \dots, x_n \end{pmatrix} dx_2 \dots dx_n \right. \\ &\quad + \sum_{k=2}^{n-1} \int_{-\infty}^t \cdots \int_{-\infty}^t K \begin{pmatrix} x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n \\ x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n \end{pmatrix} dx_1 \dots dx_{k-1} dx_{k+1} \dots dx_n \\ &\quad \left. + \int_{-\infty}^t \cdots \int_{-\infty}^t K \begin{pmatrix} x_1, x_2, \dots, x_{n-1}, t \\ x_1, x_2, \dots, x_{n-1}, t \end{pmatrix} dx_1 \dots dx_{n-1} \right\}. \end{aligned}$$

Clearly, each term in the series is continuous in  $t$  by (K2) and (K3). By using Hadamard's inequality, the right-hand side is bounded by

$$\sum_{n=1}^{\infty} \frac{|\mu|^n}{n!} n^{\frac{1}{2}n} \left( \frac{M(t)}{t} \right)^n n t^{n-1} = \frac{1}{t} \sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}n}}{(n-1)!} (|\mu| M(t))^n.$$

The series on the right-hand side converges uniformly on a compact subset of  $(\mu, t) \in \mathbb{C} \times (0, \tau)$ . Hence  $d(\mu; t)$  is continuously differentiable in  $t$  (with no exceptional points).

Successively, we prove (ii). We have

$$\frac{\partial}{\partial t} D(x, t; \mu; t) = D_y(x, t; \mu; t) + D_t(x, t; \mu; t),$$

where  $D_y$  (resp.  $D_t$ ) means the partial derivative with respect to the second (resp. the fourth) variable. We have

$$\begin{aligned} \frac{\partial}{\partial x} R(x, y; \mu; t) &= K'(x + y) + \mu \int_{-\infty}^t K'(x + z) R(z, y; \mu; t) dz, \\ \frac{\partial}{\partial y} R(x, y; \mu; t) &= K'(x + y) + \mu \int_{-\infty}^t K'(z + y) R(x, z; \mu; t) dz \end{aligned} \quad (3.19)$$

by differentiating (3.16). The integrals on the right hand sides are continuous function of  $(x, y)$  on  $\Omega_t$  by (K2), (K3) and (K4), since  $R(x, y; \mu; t)$  is continuous in  $(x, y)$ . Therefore,  $D(x, y; \mu; t)$  is continuously differentiable with respect to  $x$  and  $y$  unless  $x + y \in \Lambda$  by (K4) and (3.15). Thus  $D_y(x, t; \mu; t)$  is continuous in  $t$  except for points in  $\{\lambda - x \mid \lambda \in \Lambda\}$

for fixed  $x$ . In addition, (3.19) shows that  $|D_y(x, t; \mu; t)|$  is integrable on  $(-\infty, t]$  with respect to  $x$  by (K4). On the other hand, by definition (3.12) and (3.14),

$$\begin{aligned} \frac{\partial}{\partial t} D(x, y; \mu; t) &= -\mu K(t, t) K(x, y) + \mu K(x, t) K(t, y) \\ &+ \sum_{n=2}^{\infty} \frac{(-\mu)^n}{n!} \left\{ \int_{-\infty}^t \cdots \int_{-\infty}^t K \begin{pmatrix} x, t, x_2, \dots, x_n \\ y, t, x_2, \dots, x_n \end{pmatrix} dx_2 \dots dx_n \right. \\ &+ \sum_{k=2}^{n-1} \int_{-\infty}^t \cdots \int_{-\infty}^t K \begin{pmatrix} x, x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n \\ y, x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n \end{pmatrix} dx_1 \dots dx_{k-1} dx_{k+1} \dots dx_n \\ &\left. + \int_{-\infty}^t \cdots \int_{-\infty}^t K \begin{pmatrix} x, x_1, \dots, x_{n-1}, t \\ y, x_1, \dots, x_{n-1}, t \end{pmatrix} dx_1 \dots dx_{n-1} \right\}. \end{aligned}$$

Clearly, each term in the series is continuous in  $(x, y, t)$  by (K2) and (K3). By the row expansion of the determinant and Hadamard's inequality, the right hand side is bounded by

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{|\mu|^n}{n!} n^{\frac{1}{2}n} \left( \frac{M(t)}{t} \right)^n \int_{-\infty}^t \cdots \int_{-\infty}^t \sum_{j=1}^n |K(x, x_j)| dx_1 \dots dx_{n-1} \\ &= \frac{1}{t^2} \int_{-\infty}^t |K(x, x_1)| dx_1 \sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}n}}{(n-1)!} (|\mu| M(t))^n. \end{aligned}$$

when  $y \leq t$ . The series on the right-hand side converges uniformly on a compact subset of  $(\mu, t) \in \mathbb{C} \times (0, \tau)$ . Thus  $D_t(x, t; \mu; t)$  is continuous in  $t$ . In addition, the right-hand side shows that  $|D_t(x, t; \mu; t)|$  is integrable on  $(-\infty, t]$  with respect to  $x$ .

Hence  $\frac{\partial}{\partial t} D(x, t; \mu; t)$  is continuous on  $t \in (0, \tau) \setminus \{\lambda - x \mid \lambda \in \Lambda\}$  for fixed  $x$ , and  $|\frac{\partial}{\partial t} D(x, t; \mu; t)|$  is integrable on  $(-\infty, t]$  with respect to  $x$ .  $\square$

**3.5. The first order differential system.** As in the previous section, we assume that  $t \in [0, \tau)$  under (K1)~(K5). Then  $\phi^\varepsilon(t, x)$  is continuously differentiable with respect to  $t$  and  $x$  outside a discrete subset of  $[0, \tau) \times \mathbb{R}$ . Under this situation, we derive a first order differential system arising from  $\phi^\varepsilon(t, x)$ ,  $\varepsilon \in \{\pm 1\}$ , start from (3.8). We often denote the extended solutions  $\phi^\varepsilon(t, x)$  by  $\phi_t^\varepsilon(x)$  for the simplicity of writing.

First, we operate  $\partial/\partial t$  on both sides of (3.8). Then,

$$\begin{aligned} \frac{\partial}{\partial t} \phi_t^\varepsilon(x) + \varepsilon \frac{\partial}{\partial t} \int_{-\infty}^t K(x+y) \phi_t^\varepsilon(y) dy &= \frac{\partial}{\partial t} K(x+t); \\ \frac{\partial}{\partial t} \phi_t^\varepsilon(x) + \varepsilon \phi_t^\varepsilon(t) K(x+t) + \varepsilon \int_{-\infty}^t K(x+y) \frac{\partial}{\partial t} \phi_t^\varepsilon(y) dy &= \frac{\partial}{\partial t} K(x+t); \\ \frac{\partial}{\partial t} \phi_t^\varepsilon(x) + \varepsilon \int_{-\infty}^t K(x+y) \frac{\partial}{\partial t} \phi_t^\varepsilon(y) dy &= -\varepsilon \phi_t^\varepsilon(t) K(x+t) + \frac{\partial}{\partial t} K(x+t). \end{aligned} \quad (3.20)$$

Second, we operate  $\partial/\partial x$  on both sides of (3.8):

$$\frac{\partial}{\partial x} \phi_t^\varepsilon(x) + \varepsilon \frac{\partial}{\partial x} \int_{-\infty}^t K(x+y) \phi_t^\varepsilon(y) dy = \frac{\partial}{\partial x} K(x+t) = \frac{\partial}{\partial t} K(x+t).$$

Using the identity  $\frac{\partial}{\partial x} K(x+y) = \frac{\partial}{\partial y} K(x+y)$  and then applying integration by parts to the integral of the left-hand side, we obtain

$$\frac{\partial}{\partial x} \phi_t^\varepsilon(x) - \varepsilon \int_{-\infty}^t K(x+y) \frac{\partial}{\partial y} \phi_t^\varepsilon(y) dy = -\varepsilon \phi_t^\varepsilon(t) K(x+t) + \frac{\partial}{\partial t} K(x+t). \quad (3.21)$$

Subtracting (3.21) with choice  $-\varepsilon$  from (3.20) with  $\varepsilon$ , we obtain

$$\left\{ \frac{\partial}{\partial t} \phi_t^\varepsilon(x) - \frac{\partial}{\partial x} \phi_t^{-\varepsilon}(x) \right\} + \varepsilon \int_{-\infty}^t K(x+y) \left\{ \frac{\partial}{\partial t} \phi_t^\varepsilon(y) - \frac{\partial}{\partial y} \phi_t^{-\varepsilon}(y) \right\} dy \quad (3.22)$$

$$= -\varepsilon(\phi_t^+(t) + \phi_t^-(t))K(x+t)$$

Put

$$\mu(t) = \phi^+(t, t) + \phi^-(t, t). \quad (3.23)$$

Then  $\mu(t)$  is a real-valued function on  $[0, \tau)$ , since  $\phi^\pm(t, x)$  are real-valued. By (3.7), (3.18) and Lemmas 3.5 and 3.6,  $\mu(t)$  is continuous on  $[0, \tau)$ , satisfies  $\lim_{t \rightarrow 0^+} \mu(t) = 0$ , and is continuously differentiable on  $(0, \tau) \setminus \{\lambda/2 \mid \lambda \in \Lambda\}$ .

Equality (3.22) shows that  $\frac{\partial}{\partial t} \phi_t^\varepsilon(x) - \frac{\partial}{\partial x} \phi_t^{-\varepsilon}(x)$  is a multiple of the solution of (3.8). Hence, by comparing (3.8) with (3.22), we obtain

$$\frac{\partial}{\partial t} \phi_t^\varepsilon(x) - \frac{\partial}{\partial x} \phi_t^{-\varepsilon}(x) = -\varepsilon \mu(t) \phi_t^\varepsilon(x) \quad (\varepsilon \in \{\pm 1\}) \quad (3.24)$$

by the uniqueness of solutions (Lemmas 3.3 and 3.4). We often use (3.24) in the form

$$\left( \frac{\partial}{\partial t} + \varepsilon \mu(t) \right) \phi_t^\varepsilon(x) = \frac{\partial}{\partial x} \phi_t^{-\varepsilon}(x) \quad (\varepsilon \in \{\pm 1\}). \quad (3.25)$$

**3.6. Construction of the canonical system associated with  $E$ .** We suppose (K1)~(K5) as well as previous subsections.

**Lemma 3.7.** *For fixed  $t \in [0, \tau)$ ,  $\phi^\pm(t, x) = 0$  for  $x < -t$  and  $\phi^\pm(t, x) \ll e^{cx}$  as  $x \rightarrow +\infty$  for  $c > 0$  in (K2).*

*Proof.* We have  $\phi^\pm(t, x) = 0$  if  $x < -t$  by (K3) and (3.7). The estimate  $K(x) \ll e^{cx}$  in (K2) and (3.7) imply  $\phi^\pm(t, x) \ll e^{cx}$  as  $x \rightarrow +\infty$ .  $\square$

Now we introduce two special functions

$$\begin{aligned} \mathfrak{F}(t, x) &:= \frac{1}{2} \left( \varrho(x-t) + \int_t^\infty \varrho(x-y) \phi^+(t, y) dy \right), \\ \mathfrak{G}(t, x) &:= \frac{1}{2} \left( \varrho(x-t) - \int_t^\infty \varrho(x-y) \phi^-(t, y) dy \right) \end{aligned} \quad (3.26)$$

defined for  $(t, x) \in [0, \tau) \times \mathbb{R}$  and define  $\mathfrak{A}(t, z)$  and  $\mathfrak{B}(t, z)$  by Fourier integrals

$$\begin{aligned} \mathfrak{A}(t, z) &:= \int_{-\infty}^\infty \mathfrak{F}(t, x) e^{izx} dx = \frac{1}{2} E(z) \left( e^{izt} + \int_t^\infty \phi^+(t, x) e^{izx} dx \right), \\ -i\mathfrak{B}(t, z) &:= \int_{-\infty}^\infty \mathfrak{G}(t, x) e^{izx} dx = \frac{1}{2} E(z) \left( e^{izt} - \int_t^\infty \phi^-(t, x) e^{izx} dx \right). \end{aligned} \quad (3.27)$$

The right-hand sides of (3.27) analytic functions on the upper half plane  $\Im(z) > c$  for any fixed  $t \in [0, \tau)$  by (K1) and Lemma 3.7.

**Proposition 3.1.** *We have*

$$\begin{aligned} \mathfrak{F}(t, x) &= \frac{1}{2}(\varrho(x-t) + \varrho(-x-t)) - \frac{1}{2} \int_{-\infty}^t (\varrho(x-y) + \varrho(-x-y)) \phi^+(t, y) dy \\ &= \frac{1}{2} \int_{-\infty}^\infty (\varrho(x-y) + \varrho(-x-y)) \phi^+(t, y) dy, \\ \mathfrak{G}(t, x) &= \frac{1}{2}(\varrho(x-t) - \varrho(-x-t)) + \frac{1}{2} \int_{-\infty}^t (\varrho(x-y) - \varrho(-x-y)) \phi^-(t, y) dy \\ &= -\frac{1}{2} \int_{-\infty}^\infty (\varrho(x-y) - \varrho(-x-y)) \phi^-(t, y) dy. \end{aligned} \quad (3.28)$$

In particular,  $\mathfrak{F}(t, x)$  is even and  $\mathfrak{G}(t, x)$  is odd in  $x$ . Moreover,  $(\mathfrak{F}(t, x), \mathfrak{G}(t, x))$  are real-valued and

$$|\mathfrak{F}(t, x)|, |\mathfrak{G}(t, x)| \ll e^{-n|x|} \quad \text{for any } n > 0,$$

where the implied constant depending on  $t$ .

*Proof.*  $\mathfrak{F}(t, x)$  and  $\mathfrak{G}(t, x)$  are real-valued by definition (3.26), since  $\varrho(x)$  and  $\phi^\pm(t, x)$  are real-valued. Equalities (3.28) are obtained easily by applying (3.9) to definition (3.26). We have

$$\left| \int_{-\infty}^t (\varrho(x-y) + \varrho(-x-y)) \phi^\pm(t, y) dy \right| \ll e^{-n|x|/2} \int_{-t}^t e^{c|y|} dy$$

if  $|x| \gg t$  by (K1). Hence the first equalities of (3.28) implies the assertion.  $\square$

**Corollary 3.1.**  $\mathfrak{A}(t, z)$  is extended to an even real entire function and  $\mathfrak{B}(t, z)$  is extended to an odd real entire function as functions of  $z$  for any fixed  $t \in [0, \tau)$ .

*Proof.* This is an immediate consequence of Proposition 3.1 and (3.27).  $\square$

We denote by  $\Phi^\varepsilon(t, z)$  the Fourier transforms

$$\Phi^\varepsilon(t, z) = \int_{-\infty}^{\infty} \phi^\varepsilon(t, x) e^{izx} dx. \quad (\varepsilon \in \{\pm 1\})$$

They are defined for  $\Im(z) > c$  by Lemma 3.7 and extends to meromorphic functions on  $\mathbb{C}$  by Corollary 3.1. Moreover, we have formulas

$$\begin{aligned} \Phi^+(t, z) &= 2 \frac{\mathfrak{A}(t, z)}{E(z)} - e^{izt} + \int_{-t}^t \phi^+(t, x) e^{izx} dx, \\ \Phi^-(t, z) &= 2i \frac{\mathfrak{B}(t, z)}{E(z)} + e^{izt} + \int_{-t}^t \phi^-(t, x) e^{izx} dx \end{aligned}$$

from (3.27). By the second equations in (3.27) and parities of  $\mathfrak{A}(t, z)$  and  $\mathfrak{B}(t, z)$  as functions of  $z$ , we have

$$\begin{aligned} \mathfrak{A}(t, z) &= \frac{1}{2} \left( E(-z) e^{-izt} + E(-z) \int_t^\infty \phi^+(t, x) e^{-izx} dx \right), \\ -i\mathfrak{B}(t, z) &= -\frac{1}{2} \left( E(-z) e^{-izt} - E(-z) \int_t^\infty \phi^-(t, x) e^{-izx} dx \right). \end{aligned}$$

Substituting the left-hand sides of (3.10) to  $E(-z) e^{-izt}$  of the right-hand sides with  $E^\sharp(z) = E(-z)$ , and then noting that  $\phi^\pm(t, x) = 0$  for  $x < -t$ , we obtain

$$\begin{aligned} \mathfrak{A}(t, z) &= \frac{1}{2} (E(z) \Phi^+(t, z) + E(-z) \Phi^+(t, -z)), \\ -i\mathfrak{B}(t, z) &= -\frac{1}{2} (E(z) \Phi^-(t, z) - E(-z) \Phi^-(t, -z)), \end{aligned} \tag{3.29}$$

where we understand  $\Phi^\pm(t, z)$  as extended meromorphic functions on  $\mathbb{C}$ .

**Proposition 3.2.**  $\mathfrak{F}(t, x)$  and  $\mathfrak{G}(t, x)$  are continuous and continuously differentiable functions in both variables. In addition, they satisfy partial differential equations

$$\begin{cases} \left( \frac{\partial}{\partial t} + \mu(t) \right) \mathfrak{F}(t, x) = -\frac{\partial}{\partial x} \mathfrak{G}(t, x), \\ \left( \frac{\partial}{\partial t} - \mu(t) \right) \mathfrak{G}(t, x) = -\frac{\partial}{\partial x} \mathfrak{F}(t, x) \end{cases} \tag{3.30}$$

for  $(t, x) \in (0, \tau) \times \mathbb{R}$ .

*Proof.*  $\mathfrak{F}(t, x)$  and  $\mathfrak{G}(t, x)$  are continuous and continuously differentiable functions in both variables by (3.24) and Lemmas 3.5 and 3.6. By definition (3.26),

$$\begin{aligned} & 2 \left( \frac{\partial}{\partial t} + \mu(t) \right) \mathfrak{F}(t, x) \\ &= -\varrho'(x-t) - \varrho(x-t)\phi^+(t, t) + \mu(t)\varrho(x-t) + \int_t^\infty \varrho(x-y) \left( \frac{\partial}{\partial t} + \mu(t) \right) \phi^+(t, y) dy \\ &= -\varrho'(x-t) - \varrho(x-t)\phi^+(t, t) + \mu(t)\varrho(x-t) + \int_t^\infty \varrho(x-y) \frac{\partial}{\partial y} \phi^-(t, y) dy \\ &= -\varrho'(x-t) + \frac{\partial}{\partial x} \int_t^\infty \varrho(x-y)\phi^-(t, y) dy = -2 \frac{\partial}{\partial x} \mathfrak{G}(t, x), \end{aligned}$$

where we used (3.25) in the second equation. Hence we obtain the first line of (3.30). The second line of (3.30) is proved in a similar way.  $\square$

**Proposition 3.3.** *We have*

$$\mathfrak{F}(0, x) = \frac{1}{2}(\varrho(x) + \varrho(-x)), \quad \mathfrak{G}(0, x) = \frac{1}{2}(\varrho(x) - \varrho(-x)), \quad (3.31)$$

where  $\varrho$  is the function in (K1).

*Proof.* We have  $\Phi^\pm(0, z) = E^\sharp(z)/E(z)$  by (3.10), since  $\phi^\pm(0, x) = 0$  for negative  $x$ . Therefore,

$$\int_{-\infty}^\infty \mathfrak{F}(0, x) e^{izx} dx = \frac{1}{2}(E(z) + E^\sharp(z)), \quad \int_{-\infty}^\infty \mathfrak{G}(0, x) e^{izx} dx = \frac{1}{2}(E(z) - E^\sharp(z))$$

by (3.27) and (3.29). By taking the Fourier inverse transform, we obtain (3.31).  $\square$

We define

$$m(t) := \exp \left( \int_0^t \mu(s) ds \right) \quad (3.32)$$

and

$$H(t) := \begin{bmatrix} 1/\gamma(t) & 0 \\ 0 & \gamma(t) \end{bmatrix}, \quad \gamma(t) = m(t)^2 \quad (3.33)$$

Then  $m(t)$  is a continuous positive real-valued function on  $[0, \tau)$ ,  $m(0) = 1$  and continuously differentiable outside a discrete subset by properties of  $\mu(t)$ . Thus  $H(t)$  is a Hamiltonian on  $[0, \tau)$  consisting of continuous functions such that it has no  $H$ -indivisible intervals, that is, all points of  $[0, \tau)$  are regular. Hence it defines a canonical system on  $[0, \tau)$ . The function  $m(t)$  also has the description in terms of the operator  $K[t]$ :

$$\phi^\pm(t, t) = \pm \frac{d}{dt} \log \det(1 \pm K[t]). \quad (3.34)$$

This formula is proved in a way similar to the proof of Theorem 12 of Chapter 24 in [31]. In fact, this is a well-known formula for an integral operator defined on a finite interval with a continuous kernel. The formula (3.34) implies

$$m(t) = \frac{\det(1 + K[t])}{\det(1 - K[t])} \quad (3.35)$$

for  $t \in (0, \tau)$  by definition (3.23) and (3.32). This also holds for  $t \leq 0$  if we define  $m(t) = 1$  for  $t \leq 0$ , since  $K[t] = 0$  for  $t \leq 0$  and  $m(0) = 1$ .

Using  $m(t)$ , we define

$$F(t, x) = m(t) \cdot \mathfrak{F}(t, x), \quad G(t, x) = m(t)^{-1} \cdot \mathfrak{G}(t, x). \quad (3.36)$$

Then they satisfy the pair of partial differential equations

$$\begin{cases} \frac{\partial}{\partial t} F(t, x) + \gamma(t) \frac{\partial}{\partial x} G(t, x) = 0, \\ \frac{\partial}{\partial t} G(t, x) + \frac{1}{\gamma(t)} \frac{\partial}{\partial x} F(t, x) = 0 \end{cases} \quad (3.37)$$

by (3.30). In addition, we define

$$A(t, z) = m(t) \mathfrak{A}(t, z), \quad B(t, z) = m(t)^{-1} \mathfrak{B}(t, z) \quad (3.38)$$

and

$$E(t, z) = A(t, z) - iB(t, z). \quad (3.39)$$

Then we have

$$A(t, z) = \int_{-\infty}^{\infty} F(t, x) \cos(zx) dx, \quad B(t, z) = - \int_{-\infty}^{\infty} G(t, x) \sin(zx) dx \quad (3.40)$$

for  $t \in [0, \tau)$  and  $z \in \mathbb{C}$  by (3.27) and (3.36).  $A(t, z)$  and  $B(t, z)$  are entire functions, since  $\mathfrak{A}(t, z)$  and  $\mathfrak{B}(t, z)$  are entire functions by Proposition 3.1.

Moreover, we can verify that (3.31) and (3.37) imply that  ${}^t(A(t, z), B(t, z))$  satisfies the canonical system associated with  $H(t)$  on  $[0, \tau)$  with

$$A(0, z) = A(z), \quad B(0, z) = B(z) \quad (3.41)$$

by elementary ways. To summarize the above discussion, we obtain the following.

**Theorem 3.1.** *Let  $E$  be an entire function satisfying (K1)~(K5), and define  $A$  and  $B$  by (1.4). Let  $H$  be the Hamiltonian on  $[0, \tau)$  defined by (3.33) for (3.32) or (3.35). Let  $A(t, z)$  and  $B(t, z)$  be functions defined by (3.27) and (3.38). Then,*

- (1)  $A(t, z)$  and  $B(t, z)$  are real entire functions of  $z$  for every fixed  $t \in [0, \tau)$ ,
- (2)  $A(t, z)$  and  $B(t, z)$  are continuous and piecewise continuously differentiable function of  $t$  for every  $z \in \mathbb{C}$ ,
- (3)  $A(t, -z) = A(t, z)$  and  $B(t, z) = -B(t, z)$  for every fixed  $t \in [0, \tau)$ ,
- (4)  ${}^t(A(t, z), B(t, z))$  solves the canonical system (2.11) on  $[0, \tau)$  associated with  $H$ ,
- (5)  $A(z) = A(0, z)$ ,  $B(z) = B(0, z)$  and  $E(z) = A(0, z) - iB(0, z)$ .

The Hamiltonian  $H$  of Theorem 3.1 is extended to the Hamiltonian on  $(-\infty, \tau)$  by defining  $m(t) = 1$  for  $t < 0$ . Then, if we extend the solution  ${}^t(A(t, z), B(t, z))$  to the solution on  $(-\infty, \tau)$  by defining

$$\begin{aligned} A(t, z) &= \frac{1}{2} \left( E(z) e^{itz} + E^\sharp(z) e^{-itz} \right) = A(z) \cos(tz) + B(z) \sin(tz), \\ B(t, z) &= \frac{i}{2} \left( E(z) e^{itz} - E^\sharp(z) e^{-itz} \right) = -A(z) \sin(tz) + B(z) \cos(tz), \end{aligned}$$

it solves the canonical system (2.11) on  $(-\infty, \tau)$  associated with extended  $H$ .

**3.7. A sufficient condition for the Hermite-Biehler property.** Here is independent of the other parts of Section 3.

**Proposition 3.4.** *Let  $\gamma$  be a positive-valued continuous function defined on  $[0, \infty)$ . Suppose that  ${}^t(A(t, z), B(t, z))$  solves the canonical system on  $[0, \infty)$  associated to  $H(t) = \text{diag}(1/\gamma(t), \gamma(t))$  and that  $J(t; z, w)$  defined by (1.6) satisfies  $\lim_{t \rightarrow \infty} J(t; z, z) = 0$  for every  $z \in \mathbb{C}_+$ . Then  $E(t, z) = A(t, z) - iB(t, z)$  belongs to  $\overline{\mathbb{H}\mathbb{B}}$  for every  $t \geq 0$  (Note that every point of  $[0, \infty)$  is regular for  $H(t) = \text{diag}(1/\gamma(t), \gamma(t))$ ).*

*Proof.* We have

$$J(t; z, w) - J(s; z, w) = \frac{1}{\pi} \int_t^s \overline{A(t, z)} A(t, w) \frac{1}{\gamma(t)} dt + \frac{1}{\pi} \int_t^s \overline{B(t, z)} B(t, w) \gamma(t) dt \quad (3.42)$$



for any  $0 \leq t < s < \infty$  in a way similar to the proof of Lemma 2.1 of [16]. Taking  $w = z \in \mathbb{C}_+$  and then tending  $s$  to  $\infty$ , we have

$$J(t; z, z) = \frac{1}{\pi} \int_t^\infty |A(t, z)|^2 \frac{1}{\gamma(t)} dt + \frac{1}{\pi} \int_t^\infty |B(t, z)|^2 \gamma(t) dt$$

by  $\lim_{t \rightarrow \infty} J(t; z, z) = 0$ . On the other hand, we have

$$J(t; z, w) = \frac{\overline{E(t, z)}E(t, w) - \overline{E^\sharp(t, z)}E^\sharp(t, w)}{2\pi i(\bar{z} - w)}$$

by substituting (1.4) into (1.6). Hence we obtain

$$\frac{|E(t, z)|^2 - |E^\sharp(t, z)|^2}{4\pi \operatorname{Im} z} = \frac{1}{\pi} \int_t^\infty |A(t, z)|^2 \frac{1}{\gamma(t)} dt + \frac{1}{\pi} \int_t^\infty |B(t, z)|^2 \gamma(t) dt > 0.$$

This shows  $E(t, z)$  satisfy (1.3) and hence belongs to  $\overline{\mathbb{H}\mathbb{B}}$ .  $\square$

The above argument shows that  $\gamma(t)^{-1/2}A(t, z)$  and  $\gamma(t)^{1/2}B(t, z)$  belong to  $L^2(0, \infty)$ .

#### 4. PROOF OF THEOREM 2.2

In this section, we prove that  $E_L^{\omega, \nu}$  of (2.4) satisfies (K1)~(K5) in Section 3 (Propositions 4.1 and 4.2). Then we obtain Theorem 2.2 as the consequence of Theorem 3.1.

##### 4.1. Analytic properties of $E_L^{\omega, \nu}$ and $\Theta_L^{\omega, \nu}$ .

**Lemma 4.1.** *The entire function  $E_L^{\omega, \nu}$  define by (2.4) for  $L \in \mathcal{S}_{\mathbb{R}}$  and  $(\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$  satisfies (K1). More precisely, if we define*

$$\varrho_L^{\omega, \nu}(x) = \frac{1}{2\pi} \int_{\Im(z)=c} E_L^{\omega, \nu}(z) e^{-izx} dz \quad (4.1)$$

by taking some  $c \in \mathbb{R}$ , then it is a real-valued function in  $C^\infty(\mathbb{R})$  satisfying the estimate  $|\varrho_L^{\omega, \nu}(x)| \ll_n e^{-n|x|}$  for every  $n \in \mathbb{N}$  such that

$$E_L^{\omega, \nu}(z) = (\mathcal{F}\varrho_L^{\omega, \nu})(z) = \int_{-\infty}^\infty \varrho_L^{\omega, \nu}(x) e^{izx} dx$$

holds for all  $z \in \mathbb{C}$ .

*Proof.* From (S1),  $L(s)$  is bounded in  $\Re(s) \geq 2$ . From (S2), (S3) and the Phragmén–Lindelöf principle,  $(s-1)^{m_L}L(s)$  is bounded by a polynomial of  $s$  in any vertical strip of finite width. Thus,  $\xi_L(s)$  decays faster than  $|s|^{-n}$  for any  $n \in \mathbb{N}$  in any vertical strip of finite width. Hence  $\varrho_L^{\omega, \nu}$  is defined by (4.1) independent of  $c$  and belongs to  $C^\infty(\mathbb{R})$ . From the second equation of (2.3), we have  $(E_L^{\omega, \nu})^\sharp(z) = E_L^{\omega, \nu}(-z)$ . Thus, by taking  $c = 0$  in (4.1), we find that  $\varrho_L^{\omega, \nu}$  is real-valued. By moving the path of integration in (4.1) to the upside or downside, we see that  $|\varrho_L^{\omega, \nu}(x)| \ll_n e^{-n|x|}$  for every  $n \in \mathbb{N}$ . Taking the Fourier transform of (4.1), we obtain  $E_L^{\omega, \nu}(z) = (\mathcal{F}\varrho_L^{\omega, \nu})(z)$ .  $\square$

**Lemma 4.2.** *Let  $\Theta_L^{\omega, \nu}$  be the meromorphic function in  $\mathbb{C}$  define by (2.5) for  $L \in \mathcal{S}_{\mathbb{R}}$  and  $(\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$ . Then the estimate*

$$|\Theta_L^{\omega, \nu}(u + iv)| \ll_v |u|^{-\nu\omega d_L} \quad (4.2)$$

holds for any fixed  $v > 1/2 + \omega$ , Moreover, the estimate

$$|\Theta_L^{\omega, \nu}(u + iv)| \ll_\delta v^{-\nu\omega d_L} \quad (4.3)$$

holds uniformly for  $u \in \mathbb{R}$  and  $v \geq 1/2 + \omega + \delta$  for any fixed  $\delta > 0$ .

*Proof.* We have  $\xi_L(s) = \xi_L^\infty(s)L(s)$  by putting

$$\xi_L^\infty(s) = s^{m_L}(s-1)^{m_L} \cdot \gamma_L(s) = s^{m_L}(s-1)^{m_L} \cdot Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j). \quad (4.4)$$

Then, in order to prove (4.2) and (4.3), it is sufficient to prove

$$\left| \frac{\xi_L^\infty(s-\omega)}{\xi_L^\infty(s+\omega)} \right| \ll |u|^{-\omega d_L} \quad \text{and} \quad \left| \frac{\xi_L^\infty(s-\omega)}{\xi_L^\infty(s+\omega)} \right| \ll v^{-\omega d_L} \quad (4.5)$$

for  $s = 1/2 - i(u+iv)$  with  $v \geq 1/2 + \omega + \delta$  by (2.2) and (2.5), since  $L(s-\omega)/L(s+\omega)$  is expressed as an absolutely convergent Dirichlet series in a right-half plane  $\Re(s) > 1 + \omega$  and hence it is bounded there.

Let  $\lambda > 0$ ,  $\mu = \mu_0 + i\mu_1 \in \mathbb{C}$  with  $\mu_0 \geq 0$ . Using the Stirling formula

$$\Gamma(s) = \sqrt{\frac{2\pi}{s}} \left(\frac{s}{e}\right)^s (1 + O_\delta(|s|^{-1})) \quad |s| \geq 1, \quad |\arg s| < \pi - \delta$$

for  $\delta = \pi/4$  and  $s = 1/2 - i(u+iv)$  ( $u \in \mathbb{R}$ ,  $v > 0$ ), we have

$$\begin{aligned} \left| \frac{\Gamma(\lambda(s-\omega) + \mu)}{\Gamma(\lambda(s+\omega) + \mu)} \right| &= \sqrt{\left| 1 + \frac{2\lambda\omega}{\lambda(s-\omega) + \mu} \right|} \cdot \frac{1 + O(|\lambda(s-\omega) + \mu|^{-1})}{1 + O(|\lambda(s+\omega) + \mu|^{-1})} \cdot e^{2\lambda\omega} \\ &\quad \times \exp \left[ -2\lambda\omega \log \left| \lambda\left(\frac{1}{2} + v - \omega\right) + \mu_0 - i(\lambda u - \mu_1) \right| \right. \\ &\quad \left. + (\lambda(\frac{1}{2} + v + \omega) + \mu_0) \log \left| 1 - \frac{2\lambda\omega}{\lambda(\frac{1}{2} + v + \omega) + \mu_0 - i(\lambda u - \mu_1)} \right| \right. \\ &\quad \left. + (\lambda u - \mu_1) \arg \left( 1 - \frac{2\lambda\omega}{\lambda(\frac{1}{2} + v + \omega) + \mu_0 - i(\lambda u - \mu_1)} \right) \right]. \end{aligned}$$

In the right-hand side,

$$\begin{aligned} &\sqrt{\left| 1 + \frac{2\lambda\omega}{\lambda(s-\omega) + \mu} \right|} \cdot \frac{1 + O(|\lambda(s-\omega) + \mu|^{-1})}{1 + O(|\lambda(s+\omega) + \mu|^{-1})} \cdot e^{2\lambda\omega}, \\ &(\lambda(\frac{1}{2} + v + \omega) + \mu_0) \log \left| 1 - \frac{2\lambda\omega}{\lambda(\frac{1}{2} + v + \omega) + \mu_0 - i(\lambda u - \mu_1)} \right| \end{aligned}$$

and

$$(\lambda u - \mu_1) \arg \left( 1 - \frac{2\lambda\omega}{\lambda(\frac{1}{2} + v + \omega) + \mu_0 - i(\lambda u - \mu_1)} \right)$$

are uniformly bounded for  $u \in \mathbb{R}$  and  $v \geq 1/2 + \omega + \delta$ . Hence

$$\left| \frac{\Gamma(\lambda(s-\omega) + \mu)}{\Gamma(\lambda(s+\omega) + \mu)} \right| \ll \exp \left[ -2\lambda\omega \log \left| \lambda\left(\frac{1}{2} + v - \omega\right) + \mu_1 - i(\lambda u - \mu_2) \right| \right], \quad (4.6)$$

where the implied constant depends on  $\omega$ ,  $\lambda$ ,  $\mu$  and  $\delta > 0$ . This implies the first estimate of (4.5) by (4.4) and the definition of  $d_L$ . On the other hand, the right-hand side of (4.6) takes the maximum  $[\lambda(\frac{1}{2} + v - \omega) + \mu_1]^{-2\lambda\omega}$  at  $u = \mu_1/\lambda$  as a function of  $u$ . Hence

$$\left| \frac{\Gamma(\lambda(s-\omega) + \mu)}{\Gamma(\lambda(s+\omega) + \mu)} \right| \ll v^{-2\lambda\omega}$$

holds uniformly for  $u \in \mathbb{R}$  and  $v \geq 1/2 + \omega + \delta$ , where the implied constant depends on  $\omega$ ,  $\lambda$ ,  $\mu$  and  $\delta > 0$ . This implies the second estimate of (4.5) by (4.4) and the definition of  $d_L$ .  $\square$

Let  $a_L$  be the arithmetic function defined by the Dirichlet coefficient of  $L \in \mathcal{S}_L$ . Then, by  $a_L(1) = 1 \neq 0$ , the Dirichlet inverse  $a_L^{-1}$  exists and is given by  $a_L^{-1}(n) = -\sum_{d|n, d < n} a_L(n/d) a_L^{-1}(d)$  for  $n > 1$  and  $a_L^{-1}(1) = 1$ . We have

$$L(s)^{\pm k} = \sum_{n=1}^{\infty} \frac{a_L^{\pm k}(n)}{n^s}, \quad a_L^{\pm k} = \underbrace{a_L^{\pm 1} * \cdots * a_L^{\pm 1}}_{k \text{ times}}$$

for any positive integer  $k$ , where  $a_L^1 = a_L$  and  $*$  is the Dirichlet convolution for arithmetic functions. Using these arithmetic functions, we define

$$q_L^{\omega, \nu}(n) := n^{\omega} \sum_{d|n} \frac{a_L^{\nu}(n/d) a_L^{-\nu}(d)}{d^{2\omega}} \quad (4.7)$$

for natural numbers  $n$ .

Next we introduce the function  $g_{\omega, \lambda, \mu}$  defined on the positive real line by

$$g_{\omega, \lambda, \mu}(y) = \frac{1}{\lambda \Gamma(2\lambda\omega)} y^{\omega - \frac{1}{2} - \frac{\mu}{\lambda}} (1 - y^{-\frac{1}{\lambda}})^{2\lambda\omega - 1}$$

for  $y > 1$  and  $g_{\omega, \lambda, \mu}(y) = 0$  for  $0 < y < 1$ . Then, we define

$$\begin{aligned} \tilde{g}_L^{\omega} &= g_{\omega, \lambda_1, \mu_1} * g_{\omega, \lambda_2, \mu_2} * \cdots * g_{\omega, \lambda_r, \mu_r}, \\ \tilde{g}_L^{\omega, \nu} &= Q^{-2\nu\omega} \cdot \underbrace{\tilde{g}_L^{\omega} * \cdots * \tilde{g}_L^{\omega}}_{\nu \text{ times}} \end{aligned}$$

by using quantities  $r$ ,  $\lambda_j$ ,  $\mu_j$  and  $Q$  in (S3), where  $*$  is the multiplicative convolution  $(f * g)(x) = \int_0^{\infty} f(x/y) g(y) \frac{dy}{y}$ . In addition, using the partial fraction decomposition

$$\left( \frac{(s - \omega)(s - \omega - 1)}{(s + \omega)(s + \omega - 1)} \right)^{\nu m_L} = 1 + \sum_{k=1}^{\nu m_L} \left( \frac{X_k(\omega)}{(s + \omega - 1)^k} + \frac{Y_k(\omega)}{(s + \omega)^k} \right),$$

we define

$$r_L^{\omega, \nu}(y) = \delta_0(y) + \sum_{k=1}^{\nu m_L} \frac{1}{(k-1)!} \left( X_k(\omega) y^{1/2} + Y_k(\omega) y^{-1/2} \right) y^{-\omega} (\log y)^{k-1} \mathbf{1}_{[1, \infty)}(y)$$

for  $y > 1$  and  $r_L^{\omega, \nu}(y) = 0$  for  $0 < y < 1$ , where  $\delta_0$  is the Dirac mass at the origin.

We now define

$$g_L^{\omega, \nu} := r_L^{\omega, \nu} * \tilde{g}_L^{\omega, \nu}.$$

Then  $g_L^{\omega, \nu}$  is a  $C^\infty$ -function on  $(1, \infty)$  and vanishes on  $(0, 1)$ . The behavior of  $g_L^{\omega, \nu}$  near  $y = 1$  is singular if  $\nu > 0$  is small, but if  $\nu$  is sufficiently large with respect to  $\omega$ ,  $g_L^{\omega, \nu}$  is continuous at  $y = 1$ .

Finally, we define the function  $K_L^{\omega, \nu}$  on the real line by

$$\begin{aligned} k_L^{\omega, \nu}(y) &= \epsilon_L^{\nu} \sum_{n=1}^{\lfloor y \rfloor} \frac{q_L^{\omega, \nu}(n)}{\sqrt{n}} g_L^{\omega, \nu} \left( \frac{y}{n} \right), \\ K_L^{\omega, \nu}(x) &= k_L^{\omega, \nu}(e^x) = \epsilon_L^{\nu} \sum_{n=1}^{\lfloor \exp(x) \rfloor} \frac{q_L^{\omega, \nu}(n)}{\sqrt{n}} G_L^{\omega, \nu}(x - \log n) \end{aligned} \quad (4.8)$$

for  $x > 0$ , and  $K_L^{\omega, \nu}(x) = 0$  for  $x < 0$ , where  $G_L^{\omega, \nu}(x) = g_L^{\omega, \nu}(\exp(x))$ . The value  $K_L^{\omega, \nu}(0)$  may be undefined if  $\nu$  is small with respect to  $\omega > 0$ , but it is understood as  $K_L^{\omega, \nu}(0) = 0$  for large  $\nu$ , since  $q_L^{\omega, \nu}(1) = 1$  and  $g_L^{\omega, \nu}(1) = 0$  if  $\nu$  is large.

**Proposition 4.1.** *Define  $K_L^{\omega, \nu}$  as above for  $L \in \mathcal{S}_{\mathbb{R}}$  and  $(\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$ . Then,*

- (1)  $K_L^{\omega, \nu}$  is a continuous real-valued function on  $\mathbb{R} \setminus \{\log n \mid n \in \mathbb{N}\}$  vanishing on the negative real line  $(-\infty, 0)$ ,
- (2)  $K_L^{\omega, \nu}$  is continuously differentiable on  $\mathbb{R} \setminus \{\log n \mid n \in \mathbb{N}\}$ ,

(3) the Fourier integral formula

$$\Theta_L^{\omega,\nu}(z) = (\mathbf{F}K_L^{\omega,\nu})(z) = \int_0^\infty K_L^{\omega,\nu}(x) e^{izx} dx \quad (4.9)$$

holds for  $\Im(z) > 1/2 + \omega$  with the absolute convergence of the integral on the right-hand side. In particular,  $K_L^{\omega,\nu}$  of (4.8) coincides with the function defined in (2.6) by taking the Fourier inversion formula of (4.9).

Suppose that  $(\omega, \nu)$  satisfies the condition (2.8). Then,

- (4)  $K_L^{\omega,\nu}$  is a continuous function on  $\mathbb{R}$ ,
- (5)  $|K_L^{\omega,\nu}(x)| \ll \exp(c|x|)$  for some  $c > 0$ ,
- (6)  $|\frac{d}{dx}K_L^{\omega,\nu}(x)|$  is locally integrable.

Hence,  $E_L^{\omega,\nu}$  satisfies (K1)~(K4) under (2.8) by combining with Lemma 4.1.

Moreover, if  $\omega$  and  $\nu$  satisfy  $\nu\omega d_L > k + 1$  for some  $k \in \mathbb{N}$ ,  $K_L^{\omega,\nu}$  belongs to  $C^k(\mathbb{R})$

*Proof.* Properties (1) and (2) are trivial by definition, and (5) is a simple consequence of (2.6) and (4.2). To prove (3), we use the variable  $s = 1/2 - iz$  for convenience. If  $\omega > 0$ ,  $\lambda > 0$ ,  $\Re(\mu) \geq 0$ , we have

$$\frac{\Gamma(\lambda(s - \omega) + \mu)}{\Gamma(\lambda(s + \omega) + \mu)} = \frac{1}{\lambda\Gamma(2\lambda\omega)} \int_1^\infty y^{\omega - \frac{1}{2} - \frac{\mu}{\lambda}} (1 - y^{-\frac{1}{\lambda}})^{2\lambda\omega - 1} y^{\frac{1}{2} - s} \frac{dy}{y} \quad (4.10)$$

for  $\Re(s) > \omega - \Re(\mu)/\lambda$  by [34, (5.35) of p.195]. Therefore, we obtain

$$\int_0^\infty \tilde{g}_L^{\omega,\nu}(y) \cdot y^{\frac{1}{2} - s} \frac{dy}{y} = \left( Q^{-2\omega} \prod_{j=1}^r \frac{\Gamma(\lambda_j(s - \omega) + \mu_j)}{\Gamma(\lambda_j(s + \omega) + \mu_j)} \right)^\nu \quad (4.11)$$

for  $\Re(s) > \max_{1 \leq j \leq r} (\omega - \Re(\mu_j)/\lambda_j)$  by applying [43, Theorem 44] repeatedly to (4.10).

Applying the formula

$$\frac{1}{(k-1)!} \int_1^\infty y^{-a - \frac{1}{2}} (\log y)^{k-1} \cdot y^{\frac{1}{2} - s} \frac{dy}{y} = \frac{1}{(s+a)^k}, \quad \Re(s+a) > 0$$

to  $r_L^{\omega,\nu}$ , we have

$$\int_1^\infty r_L^{\omega,\nu}(y) \cdot y^{\frac{1}{2} - s} \frac{dy}{y} = \left( \frac{(s - \omega)(s - \omega - 1)}{(s + \omega)(s + \omega - 1)} \right)^{\nu m_L}, \quad (4.12)$$

for  $\Re(s) > 1 - \omega$  (we assumed  $\omega > 0$ ).

Then we obtain

$$\begin{aligned} \int_0^\infty g_L^{\omega,\nu}(y) \cdot y^{\frac{1}{2} - s} \frac{dy}{y} &= \int_1^\infty g_L^{\omega,\nu}(y) \cdot y^{\frac{1}{2} - s} \frac{dy}{y} \\ &= \left( \frac{(s - \omega)(s - \omega - 1)}{(s + \omega)(s + \omega - 1)} \right)^{\nu m_L} \left( Q^{-2\omega} \prod_{j=1}^r \frac{\Gamma(\lambda_j(s - \omega) + \mu_j)}{\Gamma(\lambda_j(s + \omega) + \mu_j)} \right)^\nu = \left( \frac{\xi_L^\infty(s - \omega)}{\xi_L^\infty(s + \omega)} \right)^\nu \end{aligned}$$

for  $\Re(s) > \max(1 - \omega, \max_{1 \leq j \leq r} (\omega - \Re(\mu_j)/\lambda_j))$  by applying [43, Theorem 44] to (4.11) and (4.12). On the other hand, we have

$$\begin{aligned} \left( \frac{L(s - \omega)}{L(s + \omega)} \right)^\nu &= \sum_{m=1}^\infty \frac{a_L^\nu(m) m^\omega}{m^s} \sum_{n=1}^\infty \frac{a_L^{-\nu}(n) n^{-\omega}}{n^s} \\ &= \sum_{n=1}^\infty \frac{1}{n^s} \left( n^\omega \sum_{d|n} \frac{a_L^\nu(n/d) a_L^{-\nu}(d)}{d^{2\omega}} \right) = \sum_{n=1}^\infty \frac{q_L^{\omega,\nu}(n)}{n^s} \end{aligned}$$

by definition (4.7), where the series converges absolutely for  $\Re(s) > 1 + \omega$ . By definition (4.8), we have formally

$$\begin{aligned} \int_0^\infty k_L^{\omega,\nu}(y) y^{\frac{1}{2}-s} \frac{dy}{y} &= \epsilon_L^\nu \sum_{n=1}^\infty \frac{c_L^{\omega,\nu}(n)}{n^s} \int_0^\infty g_L^{\omega,\nu}(y/n) (y/n)^{\frac{1}{2}-s} \frac{dy}{y} \\ &= \epsilon_L^\nu \left( \frac{\xi_L^\infty(s-\omega)}{\xi_L^\infty(s+\omega)} \frac{L(s-\omega)}{L(s+\omega)} \right)^\nu, \end{aligned}$$

and it is justified by Fubini's theorem for  $\Re(s) > 1 + \omega$ . By changing of variables  $y = e^x$  and  $s = 1/2 - iz$ , we obtain (4.9) and complete the proof of (3).

We prove (4) and the last line of Proposition 4.1. By (4.2),  $\Theta_L^{\omega,\nu}(u + iv)$  belongs to  $L^1(\mathbb{R})$  as a function of  $u$  if  $v$  is sufficiently large. Thus,  $K_L^{\omega,\nu}$  is uniformly continuous on  $\mathbb{R}$  by (2.6). Moreover, by (2.6) and (4.2), the formula

$$\frac{d^n}{dx^n} K_L^{\omega,\nu}(x) = \frac{1}{2\pi} \int_{\Im(z)=c} \Theta_L^{\omega,\nu}(z) (-iz)^n e^{-izx} dz$$

holds together with the absolute convergence of the integral on the right-hand side. Therefore, this shows that  $K_L^{\omega,\nu}$  is  $C^k$  if  $\nu \omega d_L > k + 1$ .

Finally, we prove (6). The derivative  $\frac{d}{dx} K_L^{\omega,\nu}$  is locally integrable by (4). On the other hand, the set of possible singularities of  $\frac{d}{dx} K_L^{\omega,\nu}$  is discrete in  $\mathbb{R}$  by (2), and  $\frac{d}{dx} K_L^{\omega,\nu}$  does not change its sign infinitely often around any possible singularity except for  $x = +\infty$  by definition of  $g_L^{\omega,\nu}$ . Therefore, the local integrability of  $\frac{d}{dx} K_L^{\omega,\nu}$  implies the local integrability of  $|\frac{d}{dx} K_L^{\omega,\nu}|$ .  $\square$

By Proposition 4.1, we find that  $E_L^{\omega,\nu}$  satisfies (K1)~(K4) if  $(\omega, \nu)$  satisfies (2.8). Successively, we show that  $E_L^{\omega,\nu}$  satisfies (K5) for sufficiently small  $\tau > 0$  unconditionally.

**4.2. Non-vanishing of Fredholm determinants: Unconditional cases.** In this section, we understand  $\mathcal{K}_L^{\omega,\nu} f$  by the integral on the right-hand side of (3.2) for  $K = K_L^{\omega,\nu}$  if it converges absolutely and locally uniform for a function  $f$ , because we do not assume that  $E_L^{\omega,\nu} \in \mathbb{H}\mathbb{B}$  (which implies that  $\mathcal{K}_L^{\omega,\nu} f$  belongs to  $L^2(\mathbb{R})$  by Lemmas 3.1 and 3.2).

**Proposition 4.2.** *Let  $L \in \mathcal{S}_{\mathbb{R}}$ . Suppose that  $(\omega, \nu)$  satisfies (2.8) and define the operator  $\mathcal{K}_L^{\omega,\nu}[t]$  on  $L^2(-\infty, t)$  by (2.9). Then, there exists  $\tau = \tau(L; \omega, \nu) > 0$  such that both  $\pm 1$  are not eigenvalues of  $\mathcal{K}_L^{\omega,\nu}[t]$  for every  $0 \leq t < \tau$ , that is, both  $1 \pm \mathcal{K}_L^{\omega,\nu}[t]$  are invertible operator on  $L^2(-\infty, t)$  for every  $0 \leq t < \tau$ . Thus  $E_L^{\omega,\nu}$  satisfies (K5) for  $[0, \tau)$ .*

*Proof.* The spectrum of  $\mathcal{K}_L^{\omega,\nu}[t]$  is discrete and consists of eigenvalues, since  $\mathcal{K}_L^{\omega,\nu}[t]$  is a Hilbert-Schmidt operator on  $L^2(-\infty, t)$  by (K2) and (K3). The statement of the proposition is equivalent that  $\mathcal{K}_L^{\omega,\nu}[t]f \neq \pm f$  for any  $f \in L^2(-\infty, t)$ , because  $1 - \mu \mathcal{K}_L^{\omega,\nu}[t]$  is invertible if  $1/\mu$  is not an eigenvalue. In addition,  $\mathcal{K}_L^{\omega,\nu}[t]f \neq \pm f$  is equivalent that  $P_t \mathcal{K}_L^{\omega,\nu} f \neq \pm f$ , since  $P_t f = f$  for  $f \in L^2(-\infty, t)$ , where  $P_t$  is the orthogonal projection from  $L^2(\mathbb{R})$  to  $L^2(-\infty, t)$ .

Suppose that  $P_t \mathcal{K}_L^{\omega,\nu} f = \pm f$  for some  $0 \neq f \in L^2(-\infty, t)$ . We have

$$(P_t \mathcal{K}_L^{\omega,\nu} f)(x) = \mathbf{1}_{(-\infty, t]}(x) \int_{-\infty}^t K_L^{\omega,\nu}(x+y) f(y) dy = 0$$

for  $-\infty < x < -t$  and  $f \in L^2(-\infty, t)$ . Therefore,  $P_t \mathcal{K}_L^{\omega,\nu} f$  is a function on  $\mathbb{R}$  having a support in  $[-t, \infty)$ , and hence the assumption implies that  $f$  has a compact support contained in  $[-t, t]$ .

We put  $g = \mathcal{K}_L^{\omega,\nu} f$ . Then, we have  $(Fg)(z) = \Theta_L^{\omega,\nu}(z)(Ff)(-z)$  for  $\Im(z) > 1/2 + \omega$  by Lemma 3.2. This means that

$$(Fg_{-v})(u) = \Theta_L^{\omega,\nu}(u + iv)(Ff_v)(-u)$$

for  $u \in \mathbb{R}$  if  $v > 1/2 + \omega$ , where we put  $g_{-v}(x) = g(x)e^{-xv}$  and  $f_v(x) = f(x)e^{xv}$ . Therefore, we have

$$\begin{aligned} \|\mathbf{F}g_{-v}\|^2 &= \|\Theta_L^{\omega,\nu}(\cdot + iv)(\mathbf{F}f_v)(-\cdot)\|^2 \\ &\leq M_v^2 \|\mathbf{F}f_v\|^2 = 2\pi M_v^2 \|f_v\|^2 = 2\pi M_v^2 \int_{-\infty}^t |f(x)|e^{2vx} dx \\ &\leq 2\pi M_v^2 e^{2vt} \int_{-\infty}^t |f(x)| dx = 2\pi M_v^2 e^{2vt} \|f\|^2, \end{aligned}$$

where  $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R})}$  and  $M_v = \max_{u \in \mathbb{R}} |\Theta_L^{\omega,\nu}(u + iv)|$ . Therefore, we have

$$\|g_{-v}\| \leq M_v e^{vt} \|f\| \quad (4.13)$$

by  $\|\mathbf{F}g_{-v}\|^2 = 2\pi \|g_{-v}\|^2$ . On the other hand, the equality  $\mathbf{P}_t \mathbf{K}_L^{\omega,\nu} f = \pm f$  implies

$$\|\mathbf{P}_t g_{-v}\|^2 = \|f_{-v}\|^2 = \int_{-\infty}^t |f(x)|^2 e^{-2vx} dx \geq e^{-2vt} \int_{-\infty}^t |f(x)|^2 dx = e^{-2vt} \|f\|^2$$

for every  $v > 0$ . Therefore,

$$e^{vt} \|\mathbf{P}_t g_{-v}\| \geq \|f\|. \quad (4.14)$$

By (4.13) and (4.14), we have  $\|g_{-v}\| \leq M_v e^{2vt} \|\mathbf{P}_t g_{-v}\|$  which implies

$$\int_{-t}^{\infty} |g(x)|^2 e^{-2vx} dx \leq M_v^2 e^{4vt} \int_{-t}^t |g(x)|^2 e^{-2vx} dx,$$

since  $g$  has a support in  $[-t, \infty)$ . By (4.3), we have

$$M_v^2 e^{4vt} \ll \exp(4vt - 2\nu\omega d_L \log v).$$

Therefore,  $M_v^2 e^{4vt} < 1$  and thus

$$\int_{-t}^{\infty} |g(x)|^2 e^{-2vx} dx < \int_{-t}^t |g(x)|^2 e^{-2vx} dx.$$

if  $t > 0$  is sufficiently small with respect to fixed  $v > 1/2 + \omega$ . This is a contradiction. Hence,  $\mathbf{P}_t \mathbf{K}_L^{\omega,\nu} f = \pm f$  is impossible for every sufficiently small  $t > 0$ .  $\square$

### 4.3. Further analytic properties of $\Theta_L^{\omega,\nu}$ .

**Proposition 4.3.** *Let  $L \in \mathcal{S}_{\mathbb{R}}$ ,  $\omega_0 \geq 0$  and  $\nu \in \mathbb{Z}_{>0}$ . Then the following statements are equivalent:*

- (1)  $L(s) \neq 0$  for  $\Re(s) > \frac{1}{2} + \omega_0$ ,
- (2)  $E_L^{\omega,\nu}$  belongs to the class  $\mathbb{H}\mathbb{B}$  for every  $\omega > \omega_0$ ,
- (3)  $\Theta_L^{\omega,\nu}$  is a meromorphic inner function in  $\mathbb{C}_+$  for every  $\omega > \omega_0$ .

The value of  $\nu$  does not affect the above equivalence.

*Proof.* Assume  $0 \leq \omega_0 \leq 1/2$ , since we have nothing to say for  $\omega > 1/2$  by Proposition 2.2 and Lemma 3.1. We find that (1) implies that  $E_L^{\omega,\nu}$  satisfies (1.3) for every  $\omega > \omega_0$  in a way similar to the proof of Proposition 2.2. If  $E_L^{\omega,\nu}(z) = \xi_L^{\omega}(\frac{1}{2} + \omega - iz)^{\nu}$  has a real zero for some  $\omega > \omega_0$ , it implies that  $L(s)$  has a zero in  $\Re(s) > \frac{1}{2} + \omega_0$ , since

$$\xi_L(\frac{1}{2} + \omega - iz) = \xi_L(\frac{1}{2} + \omega_0 - i(z + i(\omega - \omega_0))).$$

Thus  $E_L^{\omega,\nu} \in \mathbb{H}\mathbb{B}$  and we obtain (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is a consequence of Lemma 3.1. The implication (3) $\Rightarrow$ (1) is proved in a way similar to the proof of Theorem 2.3 (1) in [39].  $\square$



The value  $\omega = 1/2$  corresponds to the abscissa  $\sigma = 1$  of the absolute convergence of the Dirichlet series (2.1). The non-vanishing of  $L \in \mathcal{S}$  on the line  $\sigma = 1$  is an important problem because it relates with the analogue of the prime number theorem of  $L \in \mathcal{S}$  for example. Conrey–Ghosh [15] proved the non-vanishing of  $L \in \mathcal{S}$  on the line  $\sigma = 1$  subject to the truth of the Selberg orthogonality conjecture. Kaczorowski–Perelli [21] obtained the non-vanishing of  $L \in \mathcal{S}$  on the line  $\sigma = 1$  under a weak form of the Selberg orthogonality conjecture. As mentioned before, it is conjectured that  $\mathcal{S}$  consists only of automorphic  $L$ -functions. The non-vanishing for automorphic  $L$ -functions on the line  $\sigma = 1$  had been proved unconditionally in Jacquet–Shalika [20].

**4.4. Non-vanishing of Fredholm determinants: Conditional cases.** We suppose that  $E_L^{\omega, \nu} \in \overline{\mathbb{H}\mathbb{B}}$  throughout this subsection, otherwise it will be mentioned. Then,  $\Theta_L^{\omega, \nu}$  is inner in  $\mathbb{C}_+$  by Lemmas 3.1. This assumption is satisfied unconditionally for  $\omega > 1/2$ , and also for all  $\omega > 0$  under GRH( $L$ ) by Proposition 2.2 and Lemma 3.1 (see also Proposition 4.3). We denote by  $K_L^{\omega, \nu}$  the isometry on  $L^2(\mathbb{R})$  defined by (3.2) for  $K = K_L^{\omega, \nu}$  (cf. Lemma 3.2). It is not obvious whether the integral (3.2) for  $K = K_L^{\omega, \nu}$  defines an operator on  $L^2(\mathbb{R})$  if we do not assume that  $E_L^{\omega, \nu} \in \overline{\mathbb{H}\mathbb{B}}$ .

If  $(\omega, \nu)$  satisfies (2.8) in addition, we have

$$K_L^{\omega, \nu}[t] = P_t K_L^{\omega, \nu} P_t$$

for  $K_L^{\omega, \nu}[t]$  in (2.9) and the orthogonal projection  $P_t$  from  $L^2(\mathbb{R})$  to  $L^2(-\infty, t)$ .

**Lemma 4.3.** *Let  $L \in \mathcal{S}_{\mathbb{R}}$  and  $\omega > 0$ . Then there exists entire functions  $f_1^\omega(s)$  and  $f_2^\omega(s)$  such that they have no common zeros, satisfy*

$$\frac{\xi_L(s - \omega)}{\xi_L(s + \omega)} = \frac{f_2^\omega(s)}{f_1^\omega(s)},$$

*and the number of zeros of  $f_2^\omega(s)$  in  $|\Im(s)| \leq T$  is approximated by  $cT \log T$  for large  $T > 0$ , where  $c > 0$  is some constant.*

*Proof.* We denote by  $\mathcal{Z}_L$  the set of all zeros of  $\xi_L(s)$  and by  $m(\rho)$  the multiplicity of a zero  $\rho \in \mathcal{Z}_L$ . Then any zero of  $\xi_L(s - \omega)$  has the form  $s = \rho + \omega$  for some  $\rho \in \mathcal{Z}_L$  and has the multiplicity  $m(\rho)$ . On the other hand, if  $s = \rho + \omega$  for some  $\rho \in \mathcal{Z}_L$  and  $\xi_L(s + \omega) = 0$ , we have  $\rho + 2\omega \in \mathcal{Z}_L$ . Considering this, we set

$$\mathcal{Z}_L^\omega = \{\rho \in \mathcal{Z}_L \mid \rho + 2\omega \in \mathcal{Z}_L\},$$

and define an entire function by the Weierstrass product:

$$\begin{aligned} f_0^\omega(s) &= \prod_{\substack{\rho \in \mathcal{Z}_L^\omega \\ m(\rho) \geq m(\rho+2\omega)}} \left(1 - \frac{s}{\rho + \omega}\right)^{m(\rho+2\omega)} \exp\left(\frac{m(\rho+2\omega)s}{\rho - \omega}\right) \\ &\times \prod_{\substack{\rho \in \mathcal{Z}_L^\omega \\ m(\rho) < m(\rho+2\omega)}} \left(1 - \frac{s}{\rho + \omega}\right)^{m(\rho)} \exp\left(\frac{m(\rho)s}{\rho - \omega}\right), \end{aligned}$$

where the right-hand side converges uniformly on compact subsets in  $\mathbb{C}$ , since  $\xi_L(s)$  is an entire function of order one. In addition, we put

$$f_1^\omega(s) = \frac{\xi_L(s + \omega)}{f_0^\omega(s)}, \quad f_2^\omega(s) = \frac{\xi_L(s - \omega)}{f_0^\omega(s)}.$$

Then, by definition,  $f_1^\omega(s)$  and  $f_2^\omega(s)$  are entire functions such that they have no common zeros and satisfy

$$\frac{\xi_L(s - \omega)}{\xi_L(s + \omega)} = \frac{f_2^\omega(s)}{f_1^\omega(s)}.$$

Therefore, the remaining task is to show that  $f_2^\omega(s)$  has approximately  $cT \log T$  many zeros in  $|\Im(s)| \leq T$  for some  $c > 0$ .

We denote by  $N_L(T)$  (resp.  $N_L^\omega(T)$ ) the number of zeros in  $\mathcal{Z}_L$  (resp.  $\mathcal{Z}_L^\omega$ ) with  $|\Im(s)| \leq T$  counting with multiplicity:

$$N_L(T) = \sum_{\rho \in \mathcal{Z}_L, |\Im(s)| \leq T} m(\rho), \quad N_L^\omega(T) = \sum_{\rho \in \mathcal{Z}_L^\omega, |\Im(s)| \leq T} m(\rho)$$

and define

$$n_L^\omega(T) = \sum_{\substack{\rho \in \mathcal{Z}_L, |\Im(s)| \leq T \\ m(\rho) \geq m(\rho+2\omega)}} m(\rho+2\omega) + \sum_{\substack{\rho \in \mathcal{Z}_L, |\Im(s)| \leq T \\ m(\rho) < m(\rho+2\omega)}} m(\rho),$$

where the first sum is zero if it is an empty sum. Then,  $n_L^\omega(T) \leq N_L^\omega(T) \leq N_L(T)$  and the number of zeros of  $f_2^\omega(s)$  in  $|\Im(s)| \leq T$  is  $N_L(T) - n_L^\omega(T)$ . We recall that  $N_L(T) \sim (d_L/\pi) T \log T$ , and it is so for the number of zeros of  $\xi_L(s-\omega)$  with  $|\Im(s)| \leq T$ , where  $f(T) \sim g(T)$  means that  $f(T)/g(T) \rightarrow 1$  as  $T \rightarrow \infty$ . Therefore,  $N_L(T) - n_L^\omega(T) \sim cT \log T$  for some  $c > 0$  unless  $N_L(T) \sim n_L^\omega(T)$ .

Now we prove that  $N_L(T) \not\sim n_L^\omega(T)$  by contradiction. Suppose that  $N_L(T) \sim n_L^\omega(T)$ . Then,  $N_L(T) \sim N_L^\omega(T)$ , since  $n_L^\omega(T) \leq N_L^\omega(T) \leq N_L(T)$ . We put  $\Sigma_L^\omega = \{\rho \in \mathcal{Z}_L^\omega \mid 1-\rho \in \mathcal{Z}_L^\omega\}$  and

$$M_L^\omega(T) = \sum_{\rho \in \Sigma_L^\omega, |\Im(\rho)| \leq T} m(\rho).$$

Then,  $M_L^\omega(T) \sim N_L(T)$ , because  $N_L(T) \sim N_L^\omega(T)$  and functional equations (2.3) imply that  $\mathcal{Z}_L^\omega$  is closed under  $\rho \mapsto 1-\rho$  and  $\rho \mapsto \bar{\rho}$  except for a relatively small subset counting with multiplicity. If we take a zero  $\rho \in \Sigma_L^\omega$ , then  $1-\rho \in \mathcal{Z}_L^\omega$  by the definition of  $\Sigma_L^\omega$ , and thus  $1-\rho+2\omega \in \mathcal{Z}_L$  by the definition of  $\mathcal{Z}_L^\omega$ . Therefore,  $\rho-2\omega = 1-(1-\rho+2\omega) \in \mathcal{Z}_L$  by the first functional equation of (2.3). As a consequence,  $\rho \in \Sigma_L^\omega$  implies  $\rho-2\omega \in \mathcal{Z}_L^\omega$ . On the other hand,  $M_L^\omega(T) \sim N_L^\omega(T) \sim N_L(T)$  shows that  $\rho-2\omega \in \mathcal{Z}_L^\omega$  implies  $\rho-2\omega \in \Sigma_L^\omega$  almost surely. Taken together,  $\rho \in \Sigma_L^\omega$  implies  $\rho-2\omega \in \Sigma_L^\omega$  almost surely and this process is continued repeatedly. However, it is impossible, because all zeros of  $\xi_L(s)$  must lie in the critical strip. Hence,  $N_L(T) \not\sim n_L^\omega(T)$ .  $\square$

**Lemma 4.4.** *Let  $t \geq 0$ . Suppose that  $(\omega, \nu)$  satisfies (2.8). Then the support of  $\mathbf{K}_L^{\omega, \nu} \mathbf{P}_t f$  is not compact for every  $f \in L^2(\mathbb{R})$  unless  $\mathbf{K}_L^{\omega, \nu} \mathbf{P}_t f = 0$ .*

*Proof.* We prove this by contradiction. Suppose that  $\mathbf{K}_L^{\omega, \nu} \mathbf{P}_t f \neq 0$  and has a compact support. Then  $\mathbf{F}\mathbf{K}_L^{\omega, \nu} \mathbf{P}_t f$  is an entire function of exponential type by the Paley-Wiener theorem. On the other hand, we have

$$\mathbf{F}\mathbf{K}_L^{\omega, \nu} \mathbf{P}_t f(z) = \Theta_L^{\omega, \nu}(z) \cdot \mathbf{F}\mathbf{P}_t f(-z)$$

by Lemma 3.2. If we put  $G(z) = \mathbf{F}\mathbf{P}_t f(-z)/f_1^\omega(\frac{1}{2}-iz)^\nu$ , then we have

$$\mathbf{F}\mathbf{K}_L^{\omega, \nu} \mathbf{P}_t f(z) = f_2^\omega(\frac{1}{2}-iz)^\nu \cdot G(z),$$

where  $f_1^\omega$  and  $f_2^\omega$  are functions in Lemma 4.3. Here  $G(z)$  is entire, because, by Lemma 4.3, the zeros of the numerator  $f_2^\omega(\frac{1}{2}-iz)^\nu$  of  $\Theta_L^{\omega, \nu}(z)$  can not kill the zeros of the denominator  $f_1^\omega(\frac{1}{2}-iz)^\nu$ , which therefore must be killed by zeros of  $\mathbf{F}\mathbf{P}_t f(-z)$ . This allows  $f_2^\omega(\frac{1}{2}-iz)^\nu$  to be factored out.

The entire function on the right-hand side has at least  $cT \log T$  many zeros in the disk of radius  $T$  around the origin as  $T \rightarrow \infty$  for some  $c > 0$  by Lemma 4.3. However all entire functions of exponential type have at most  $O(T)$  zeros in the disk of radius  $T$  around the origin, as  $T \rightarrow \infty$ , because of the Jensen formula ([29, §2.5 (15)]). This is a contradiction.

As the above, it is not necessary to assume that  $\Theta$  is inner in  $\mathbb{C}_+$  for Lemma 4.4.  $\square$

**Proposition 4.4.** *Let  $t \geq 0$ . We have i)  $\mathcal{K}_L^{\omega,\nu}[t]f = 0$  for every  $f \in L^2(-\infty, -t)$ , ii)  $\|\mathcal{K}_L^{\omega,\nu}[t]f\| \neq \|f\|$  for every  $0 \neq f \in L^2(-\infty, t)$ , and iii)  $\|\mathbf{H}_{\omega,a}\| < 1$ . In particular,  $1 \pm \mathcal{K}_L^{\omega,\nu}[t]$  are invertible operator on  $L^2(-\infty, t)$  for every  $t \geq 0$ .*

*Proof.* First, we note that  $\mathcal{K}_L^{\omega,\nu}f$  is defined for every  $f \in L^2(-\infty, t)$  by Lemma 3.2 for  $K = K_L^{\omega,\nu}$ . Because  $\int_{-\infty}^t K_L^{\omega,\nu}(x+y)f(y)dy = 0$  for  $x < -t$  by (K3), we obtain i).

To prove ii), it is sufficient to show  $\|\mathcal{K}_L^{\omega,\nu}[t]f\| \neq \|f\|$  unless  $f = 0$ , because  $\|\mathcal{K}_L^{\omega,\nu}[t]\| \leq \|\mathbf{P}_t\| \cdot \|\mathcal{K}_L^{\omega,\nu}\| \cdot \|\mathbf{P}_t\| = 1$  by Lemma 3.2, and  $\|\mathcal{K}_L^{\omega,\nu}[t]f\| \leq \|\mathcal{K}_L^{\omega,\nu}[t]\| \cdot \|f\| \leq \|f\|$ . Here  $\|\mathcal{K}_L^{\omega,\nu}[t]f\| \neq \|f\|$  is equivalent to  $\|\mathbf{P}_t \mathcal{K}_L^{\omega,\nu}f\| \neq \|f\|$ , since  $\mathbf{P}_t f = f$  for  $f \in L^2(-\infty, t)$ . Suppose that  $\|\mathbf{P}_t \mathcal{K}_L^{\omega,\nu}f\| = \|f\|$  for some  $0 \neq f \in L^2(-\infty, t)$ . Then it implies  $\|\mathbf{P}_t \mathcal{K}_L^{\omega,\nu}f\| = \|\mathcal{K}_L^{\omega,\nu}f\|$  by  $\|\mathcal{K}_L^{\omega,\nu}f\| = \|f\|$ . Therefore

$$\int_{-\infty}^t |\mathcal{K}_L^{\omega,\nu}f(x)|^2 dx = \int_{-\infty}^{\infty} |\mathcal{K}_L^{\omega,\nu}f(x)|^2 dx.$$

Thus  $\mathcal{K}_L^{\omega,\nu}f(x) = 0$  for almost every  $x > t$ . On the other hand, we have

$$\mathcal{K}_L^{\omega,\nu}f(x) = \int_{-\infty}^t K_L^{\omega,\nu}(x+y)f(y)dy = \int_{-x}^t K_L^{\omega,\nu}(x+y)f(y)dy = 0$$

for  $x < -t$  by  $f \in L^2(-\infty, t)$ . Hence  $\mathcal{K}_L^{\omega,\nu}f$  has a compact support contained in  $[-t, t]$ . However, it is impossible for any  $f \neq 0$  by Lemma 4.4. As the consequence  $\|\mathcal{K}_L^{\omega,\nu}[t]f\| < \|f\|$  for every  $0 \neq f \in L^2(-\infty, t)$ .

Finally, we prove iii). As found in Section 3.1,  $\mathcal{K}_L^{\omega,\nu}[t]$  is a self-adjoint compact operator (because the Hilbert-Schmidt operator is compact). Therefore,  $\mathcal{K}_L^{\omega,\nu}[t]$  has purely discrete spectrum which has no accumulation points except for 0, and one of  $\pm \|\mathcal{K}_L^{\omega,\nu}[t]\|$  is an eigenvalue of  $\mathcal{K}_L^{\omega,\nu}[t]$ . However, by ii), every eigenvalue of  $\mathcal{K}_L^{\omega,\nu}[t]$  has an absolute value less than 1. Hence  $\|\mathcal{K}_L^{\omega,\nu}[t]\| < 1$ .  $\square$

## 5. THEORY OF DE BRANGES SPACES

In this section, we review several basic notions and properties of de Branges spaces as a preparation to the next section. General theory of de Branges spaces is given in the book [9], but the proofs of results are more accessible in de Branges' earlier papers [4, 5, 6, 7, 8]. See also [45], [46] and references there in.

**5.1. Hardy spaces.** The Hardy space  $H^2 = H^2(\mathbb{C}_+)$  in the upper half-plane  $\mathbb{C}_+$  is defined to be the space of all analytic functions  $f$  in  $\mathbb{C}_+$  endowed with norm  $\|f\|_{H^2}^2 := \sup_{v>0} \int_{\mathbb{R}} |f(u+iv)|^2 du < \infty$ . The Hardy space  $\bar{H}^2 = H^2(\mathbb{C}_-)$  in the lower half-plane  $\mathbb{C}_-$  is defined in a similar way. As usual we identify  $H^2$  and  $\bar{H}^2$  with subspaces of  $L^2(\mathbb{R})$  via nontangential boundary values on the real line such that  $L^2(\mathbb{R}) = H^2 \oplus \bar{H}^2$ . The Fourier transform provide an isometry of  $L^2(\mathbb{R})$  up to a constant such that  $H^2 = FL^2(0, \infty)$  and  $\bar{H}^2 = FL^2(-\infty, 0)$  by the Paley-Wiener theorem.

**5.2. De Branges spaces.** For  $E \in \overline{\mathbb{H}\mathbb{B}}$ , the set

$$\mathcal{B}(E) := \{f \mid f \text{ is entire, } f/E \text{ and } f^\sharp/E \in H^2\}$$

forms a Hilbert space under the norm  $\|f\|_{\mathcal{B}(E)} := \|f/E\|_{L^2(\mathbb{R})}$ . The Hilbert space  $\mathcal{B}(E)$  is called the *de Branges space* generated by  $E$ . The de Branges space  $\mathcal{B}(E)$  is a reproducing kernel Hilbert space endowed with the reproducing kernel

$$J(z, w) = \frac{\overline{E(z)}E(w) - \overline{E^\sharp(z)}E^\sharp(w)}{2\pi i(\bar{z} - w)} \quad (z, w \in \mathbb{C}_+).$$

The reproducing formula  $f(z) = \langle f, J(z, \cdot) \rangle$  for  $f \in \mathcal{B}(E)$  and  $z \in \mathbb{C}_+$  remains true for  $z \in \mathbb{R}$  if  $\Theta = E^\sharp/E$  is analytic in a neighborhood of  $z$ , where  $\langle f, g \rangle = \int_{\mathbb{R}} f(u)\overline{g(u)}du$ .

**5.3. Axiomatic characterization of de Branges spaces.** The Hilbert space  $\mathcal{H}$  consisting of entire functions forms a de Branges space if and only if it satisfies the following three axioms:

- (dB1) For each  $z \in \mathbb{C}$  the point evaluation  $\Phi \mapsto \Phi(z)$  is a continuous linear functional on  $\mathcal{H}$ .
- (dB2) If  $\Phi \in \mathcal{H}$ ,  $\Phi^\sharp$  belongs to  $\mathcal{H}$  and  $\|\Phi\|_{\mathcal{H}} = \|\Phi^\sharp\|_{\mathcal{H}}$ .
- (dB3) If  $w \in \mathbb{C} \setminus \mathbb{R}$ ,  $\Phi \in \mathcal{H}$  and  $\Phi(w) = 0$ ,

$$\frac{z - \bar{w}}{z - w} \Phi(z) \in \mathcal{H} \quad \text{and} \quad \left\| \frac{z - \bar{w}}{z - w} \Phi(z) \right\|_{\mathcal{H}} = \|\Phi\|_{\mathcal{H}}.$$

If  $\mathcal{H}$  satisfies the above axioms, there exists  $E \in \overline{\mathbb{H}\mathbb{B}}$  such that  $\mathcal{H} = \mathcal{B}(E)$  and  $\|f\|_{\mathcal{H}} = \|f\|_{\mathcal{B}(E)}$  for all  $f \in \mathcal{H}$ . The possibility of  $E$  is not unique. In fact,  $\mathcal{B}(E) = \mathcal{B}(E_\theta)$  for any  $\theta \in [0, \pi)$  and  $E_\theta(z) = e^{i\theta} E(z)$ .

**5.4. De Branges subspaces.** A subspace  $\mathcal{V}$  of  $\mathcal{B}(E)$  is called a *de Branges subspace* if it forms a de Branges space under the norm of  $\mathcal{B}(E)$ , that is, there exists  $E' \in \overline{\mathbb{H}\mathbb{B}}$  such that  $\mathcal{V} = \mathcal{B}(E')$  and  $\|f\|_{\mathcal{V}} = \|f\|_{\mathcal{B}(E)}$  for all  $f \in \mathcal{V}$ , where  $E'$  has the same real zeros as  $E$  including multiplicities.

For a given  $E \in \mathbb{H}\mathbb{B}$ , there exists a chain of de Branges subspaces  $\mathcal{B}(E_t) \subset \mathcal{B}(E)$ ,  $0 \leq t < c (\leq \infty)$ , endowed with a family of entire functions  $E_t \in \mathbb{H}\mathbb{B}$  satisfying  $E_t(0) = E(0)$  such that  $\mathcal{B}(E_t) \supset \mathcal{B}(E_s)$  if  $t < s$ , and the parametrized pair of real entire functions  $(A_t, B_t) := (\frac{1}{2}(E_t + E_t^\sharp), \frac{i}{2}(E_t - E_t^\sharp))$  solve the canonical system on  $[0, c)$  associated with some Hamiltonian  $H(t)$  and  $E_0(z) = E(z)$  (see [9, Theorem 40], but note that the result is formulated in terms of integral equations and that we need a changing of variables to apply the result.)

For  $0 \leq t < s < c$ , there exists a transfer matrix  $M(t, s; z)$  which is a matrix of entire functions having the property that

$$\begin{bmatrix} A_t(z) \\ B_t(z) \end{bmatrix} = M(t, s; z) \begin{bmatrix} A_s(z) \\ B_s(z) \end{bmatrix}$$

and  $\det M(t, s; z) = 1$  ([9, Theorem 33]).

**5.5. Model subspaces.** We review a few results on model spaces according to Havin–Mashreghi [17, 18] (and also Baranov [1]). For an inner function  $\Theta$ , a *model subspace* (or coinvariant subspace)  $\mathcal{K}(\Theta)$  is defined by the orthogonal complement

$$\mathcal{K}(\Theta) = H^2 \ominus \Theta H^2, \tag{5.1}$$

where  $\Theta H^2 = \{\Theta(z)F(z) \mid F \in H^2\}$ . It has the alternative representation

$$\mathcal{K}(\Theta) = H^2 \cap \Theta \bar{H}^2. \tag{5.2}$$

It is known that every meromorphic inner function is expressed as  $\Theta = E^\sharp/E$  by using some  $E \in \overline{\mathbb{H}\mathbb{B}}$ . If  $\Theta$  is a meromorphic inner function such that  $\Theta = E^\sharp/E$ , the model subspace  $\mathcal{K}(\Theta)$  is isomorphic and isometric to the de Branges space  $\mathcal{B}(E)$  as a Hilbert space by  $\mathcal{K}(\Theta) \rightarrow \mathcal{B}(E) : F(z) \mapsto E(z)F(z)$ .

The changing of consideration from  $\mathcal{B}(E)$  to  $\mathcal{K}(\Theta)$  has the advantage that spaces  $\Theta H^2$ ,  $\Theta \bar{H}^2$ ,  $H^2 \ominus (H^2 \cap \Theta H^2)$  and  $H^2 \cap \Theta \bar{H}^2$  are defined even if  $\Theta$  is not necessarily a meromorphic inner function in  $\mathbb{C}_+$ , and it allows us to study these spaces without assuming  $E \in \overline{\mathbb{H}\mathbb{B}}$ . Note that  $\Theta H^2 \not\subset H^2$  in general if  $\Theta$  is not necessary a inner function in  $\mathbb{C}_+$ . As developed in [10, 11, 12, 13], the Hankel type operator  $K$  with the kernel  $K(x+y)$  is quite useful to study  $\mathcal{K}(\Theta)$  and the above spaces via Fourier analysis. This fact was already used implicitly in Sections 3 and 4 and will be used in Sections 6 (and also Section 9.2).

## 6. PROOF OF THEOREM 2.3

We prove Theorem 2.3 by studying de Branges subspaces of  $\mathcal{B}(E_L^{\omega, \nu})$ .

**6.1. Formulas for reproducing kernels of de Branges subspaces.** We start from general theory for entire functions  $E \in \mathbb{H}\mathbb{B}$  satisfying (K1)~(K4). Then  $\Theta = E^\sharp/E$  is inner in  $\mathbb{C}_+$ ,  $f \mapsto Kf$  defines an isometry on  $L^2(\mathbb{R})$  satisfying  $K^2 = \text{id}$  by Lemmas 3.1 and 3.2. In addition, we assume (K5) with  $\tau = \infty$ . As proved in Section 4,  $E_L^{\omega, \nu}$  satisfies all these conditions for  $\omega > 1/2$  unconditionally and for all  $\omega > 0$  under GRH( $L$ ). Under the above setting, we study de Branges subspaces of  $\mathcal{B}(E)$ .

Let  $\mathcal{V}_t$  be the Hilbert space of all functions  $f$  such that both  $f$  and  $Kf$  are square integrable functions having supports in  $[t, \infty)$ :

$$\mathcal{V}_t = L^2(t, \infty) \cap K(L^2(t, \infty)).$$

**Lemma 6.1.** *We have  $F(\mathcal{V}_0) = \mathcal{K}(\Theta)$  and thus  $E(z)F(\mathcal{V}_0) = \mathcal{B}(E)$ .*

*Proof.* If  $f \in \mathcal{V}_0$ ,  $Ff$  and  $FKf$  belong to the Hardy space  $H^2$ . On the other hand, we have  $(FKf)(z) = \Theta(z)(Ff)(-z)$  by Lemma 3.2. This implies  $(Ff)(z) = \Theta(z)(FKf)(-z)$ , since  $\Theta(z)\Theta(-z) = 1$  by (K1). Therefore,  $Ff$  belongs to  $\mathcal{K}(\Theta)$  by (5.2). Conversely, if  $F \in \mathcal{K}(\Theta)$ , there exists  $f \in L^2(0, \infty)$  and  $g \in L^2(-\infty, 0)$  such that

$$F(z) = (Ff)(z) = \Theta(z)(Fg)(z).$$

We have  $(Fg)(-z) = \Theta(z)(Ff)(-z)$  by using  $\Theta(z)\Theta(-z) = 1$  again. Here  $(Fg)(-z) = (Fg^-)(z)$  for  $g^-(x) = g(-x) \in L^2(0, \infty)$ , and  $\Theta(z)(Ff)(-z) = (FKf)(z)$  as above. Hence  $Kf$  belongs to  $L^2(0, \infty)$ , and thus  $f \in \mathcal{V}_0$ .  $\square$

**Lemma 6.2.** *We have  $\mathcal{V}_t \neq \{0\}$  for every  $t \geq 0$ .*

*Proof.* The case of  $t = 0$  is Lemma 6.1. We suppose  $t > 0$ . By the general theory of Hilbert spaces, the orthogonal complement  $\mathcal{V}_t^\perp$  of  $\mathcal{V}_t$  in  $L^2(\mathbb{R})$  is equal to the closure of  $L^2(-\infty, t) + K(L^2(-\infty, t))$ . We show that  $L^2(-\infty, t) + K(L^2(-\infty, t))$  is closed.

We suppose that  $w = u + Kv$  for  $u, v \in L^2(-\infty, t)$ . Then we have  $P_t w = u + K[t]v$  and  $P_t K w = K[t]u + v$  by  $K^2 = \text{id}$ . It is solved as  $u = (1 - K[t]^2)^{-1}(P_t - K[t]P_t K)w$  and  $v = (1 - K[t]^2)^{-1}(P_t K - K[t]P_t)w$ , since  $\pm 1$  are not eigenvalues of  $K[t]$ . Therefore, if  $w_n = u_n + Kv_n$  is  $L^2$ -convergent, it implies that both  $u_n$  and  $v_n$  are also  $L^2$ -convergent, and hence the space  $L^2(-\infty, t) + K(L^2(-\infty, t))$  is closed. Hence we obtain

$$\mathcal{V}_t^\perp = L^2(-\infty, t) + K(L^2(-\infty, t)). \quad (6.1)$$

To prove  $\mathcal{V}_t \neq \{0\}$ , it is sufficient to show that  $L^2(-\infty, t) + K(L^2(-\infty, t))$  is a proper closed subspace of  $L^2(\mathbb{R})$ . Suppose that  $f \in K(L^2(-\infty, t))$ . Then the restriction of  $f$  to  $(t, \infty)$  is a continuous on  $(t, \infty)$ . In fact, if we put  $g(x) := (Kf)(x) = \int_{-x}^t K(x+y)f(y) dy$  for  $f \in L^2(-\infty, t)$ , the continuity of  $K(x)$  implies the continuity of  $g$  by

$$\begin{aligned} |g(x+\delta) - g(x)| &= \left| \int_{-x-\delta}^t K(x+\delta+y)f(y) dy - \int_{-x}^t K(x+y)f(y) dy \right| \\ &\leq \int_{-x-\delta}^x |K(x+\delta+y)||f(y)| dy + \int_{-x}^t |K(x+\delta+y) - K(x+y)||f(y)| dy \\ &\leq \left( \int_{-x-\delta}^x |K(x+\delta+y)|^2 dy + C_x \cdot \delta \cdot |t+x| \right) \|f\|^2 \\ &\leq \delta \left( \max_{0 \leq y \leq 3x} |K(y)|^2 + C_x \cdot \delta \cdot |t+x| \right) \|f\|^2, \end{aligned}$$

where  $0 < \delta < x$  and we used the mean value theorem and the Schwartz inequality. Hence, if  $f \in L^2(-\infty, t) + K(L^2(-\infty, t))$ ,  $f$  is continuous on  $(t, \infty)$ . Thus  $L^2(-\infty, t) + K(L^2(-\infty, t))$  is a proper closed subspace of  $L^2(\mathbb{R})$ .  $\square$

**Lemma 6.3.** *The subspace  $E(z)F(\mathcal{V}_t)$  of  $\mathcal{B}(E)$  is a de Branges subspace for every  $t > 0$ .*

*Proof.* We show that  $\mathcal{H} = E(z)\mathbf{F}(\mathcal{V}_t)$  satisfies the axioms of the de Branges spaces. It is trivial that  $\mathcal{H}$  satisfies (dB1), (dB2) and the equality of norms in (dB3), since  $\mathcal{H}$  is a subspace of  $\mathcal{B}(E)$  as a Hilbert space.

Suppose that  $w \in \mathbb{C} \setminus \mathbb{R}$ ,  $\Phi \in E(z)\mathbf{F}(\mathcal{V}_t)$  and  $\Phi(w) = 0$ , where  $\Phi(z) = E(z)(\mathbf{F}f)(z)$  for some  $f \in \mathcal{V}_t$ . Then, we have

$$\frac{z - \bar{w}}{z - w}(\mathbf{F}f)(z) = (\mathbf{F}f_w)(z)$$

for some  $f_w \in \mathcal{V}_0 (\subset L^2(0, \infty))$ , since  $\mathcal{B}(E) = E(z)\mathbf{F}(\mathcal{V}_0)$  is a de Branges space. Therefore, it is sufficient to show that  $f_w$  and  $\mathbf{K}f_w$  have support in  $[t, \infty)$ .

We have

$$f_w(x) = \frac{1}{2\pi} \int_{-\infty+ic}^{+\infty+ic} \frac{z - \bar{w}}{z - w}(\mathbf{F}f)(z) e^{-izx} dz$$

for any  $c > 0$ . On the other hand, if we put

$$g(x) = f(x) - i(w - \bar{w}) \int_0^{x-t} f(x-y) e^{-iwy} dy,$$

$g$  has a support in  $[t, \infty)$  and we have  $(\mathbf{F}g)(z) = \frac{z - \bar{w}}{z - w}(\mathbf{F}f)(z)$  for  $\Im(z) > \Im(w)$ . Therefore,

$$g(x) = \frac{1}{2\pi i} \int_{-\infty+ic'}^{+\infty+ic'} \frac{z - \bar{w}}{z - w}(\mathbf{F}f)(z) e^{-izx} dz$$

if  $c' > \Im(w)$ . Hence  $f_w = g$  and this shows that the support of  $f_w$  is in  $[t, \infty)$ .

On the other hand, we have

$$(\mathbf{K}f_w)(z) = \Theta(z)(\mathbf{F}f_w)(-z) = \Theta(z) \frac{z + \bar{w}}{z + w}(\mathbf{F}f)(-z) = \frac{z + \bar{w}}{z + w}(\mathbf{K}f)(z).$$

by  $(\mathbf{K}f)(z) = \Theta(z)(\mathbf{F}f)(-z)$ . Here  $\mathbf{K}f \in \mathcal{V}_t$  by the definition of  $\mathcal{V}_t$  and  $(\mathbf{K}f)(-w) = 0$ . Therefore, we find that there exists  $(\mathbf{K}f)_{-w} \in \mathcal{V}_0$  having the support in  $[t, \infty)$  such that  $(\mathbf{K}f)_{-w}(z) = \frac{z + \bar{w}}{z + w}(\mathbf{K}f)(z)$  in a way similar to the case of  $f$ . Clearly,  $\mathbf{K}f_w = (\mathbf{K}f)_{-w}$ . Hence  $\mathbf{K}f_w$  has the support in  $[t, \infty)$ .  $\square$

By Lemmas 6.1 and 6.2, the evaluation  $f \mapsto E(z)(\mathbf{F}f)(z)$  at  $z \in \mathbb{C}$  defines a continuous linear functional on  $\mathcal{V}_t$ . Therefore, there exists a unique vector  $X_z^t \in \mathcal{V}_t$  such that

$$\langle f, X_z^t \rangle = \int_t^\infty f(x) \overline{X_z^t(x)} dx = E(z)(\mathbf{F}f)(z)$$

holds for all  $f \in \mathcal{V}_t$  by the Riesz representation theorem.

**Theorem 6.1.** *Let  $E \in \mathbb{H}\mathbb{B}$ . Suppose that  $E$  satisfies (K1)~(K4) and (K5) for  $\tau = \infty$ . Let  $t \geq 0$  and let  $X_z^t$  be the vector in  $\mathcal{V}_t$  such that  $\langle f, X_z^t \rangle = E(z)(\mathbf{F}f)(z)$  for all  $f \in \mathcal{V}_t$ . Then, we have*

$$J(t; z, w) = \frac{1}{2\pi} \langle X_z^t, X_w^t \rangle, \quad (6.2)$$

for all  $z, w \in \mathbb{C}$ , where the left-hand side is the function in (1.6) and  $\langle f, g \rangle = \int_{\mathbb{R}} f(u) \overline{g(u)} du$ . Moreover, the entire function  $E(t, z)$  of (3.39) belongs to  $\mathbb{H}\mathbb{B}$  and

$$E(z)\mathbf{F}(\mathcal{V}_t) = \mathcal{B}(E(t, z))$$

for every  $t \geq 0$ .

*Proof.* We prove Theorem 6.1 according to the way of [13, Section 5]. We set  $Y_z^t(x) = (1/E(z))\overline{X_z^t(x)}$ . Then  $Y_z^t$  is the unique vector in  $\mathcal{V}_t$  such that  $(f, Y_z^t) = (\mathbf{F}f)(z)$  for all  $f \in \mathcal{V}_t$ , where  $(f, g) = \int_{\mathbb{R}} f(x)g(x) dx$ .

First, we consider under the restriction  $\Im(z) > c$  for  $c$  in (K2) and define  $Y_z^t$  to be the orthogonal projection of  $\mathbf{1}_{[t, \infty)}(x)e^{izx}$  to  $\mathcal{V}_t$ . Then  $\mathbf{1}_{[t, \infty)}(x)e^{izx} \in L^2(\mathbb{R})$  by  $\Im(z) > c$ , and therefore

$$g_z^t(x) = \mathbf{K}\left(\mathbf{1}_{[t, \infty)}(*)e^{iz*}\right) = \int_t^\infty K(x+y)e^{izy} dy \quad (6.3)$$



is defined. By (6.1), there are unique vectors  $u_z^t, v_z^t$  in  $L^2(-\infty, t)$  such that

$$\mathbf{1}_{[t, \infty)}(x)e^{izx} = Y_z^t(x) + u_z^t(x) + K(v_z^t)(x) \quad (6.4)$$

and they are the solutions to the system of equations

$$u_z^t + P_t K P_t(v_z^t) = 0, \quad P_t K P_t(u_z^t) + v_z^t = P_t(g_z^t),$$

since  $K^2 = \text{id}$  and  $KX_z^t \in L^2(t, \infty)$  by Lemmas 3.1 and 3.2. From the first (resp. second) equation, we see that  $u_z^t$  (resp.  $v_z^t$ ) is the restriction to  $(-\infty, t)$  of an function defined on  $\mathbb{R}$ . In fact, we have  $u_z^t = P_t U_z^t$  and  $v_z^t = P_t V_z^t$  if we define  $U_z^t = -K P_t(v_z^t)$  and  $V_z^t = g_z^t - K P_t(u_z^t)$ . Moreover,  $U_z^t$  and  $V_z^t$  are also defined as the solution of the equation

$$U_z^t + K P_t(V_z^t) = 0, \quad K P_t(U_z^t) + V_z^t = g_z^t \quad (6.5)$$

on  $\mathbb{R}$ , and the right hand side of (6.4) becomes

$$= Y_z^t(x) + \mathbf{1}_{(-\infty, t)}(x)U_z^t(x) + K P_t(V_z^t)(x) = Y_z^t(x) - \mathbf{1}_{[t, \infty)}(x)U_z^t(x).$$

Hence (6.4) becomes

$$Y_z^t(x) = \mathbf{1}_{[t, \infty)}(x)(e^{izx} + U_z^t(x)). \quad (6.6)$$

Let  $D_z = i\frac{d}{dx} + z$ . This annihilates  $e^{izx}$  and satisfies the relation  $D_z K = -K D_{-z}$ .

Applying  $D_z$  to the left-hand side of the first equation of (6.5) and using  $\frac{d}{dx}P_t(V_z^t) = P_t(\frac{d}{dx}V_z^t) - V_z^t(t)\delta_t(x)$ ,

$$\begin{aligned} D_z U_z^t + D_z K P_t(V_z^t) &= D_z U_z^t - K D_{-z} P_t(V_z^t) \\ &= D_z U_z^t - K(P_t D_{-z}(V_z^t) - iV_z^t(t)\delta_t(x)). \end{aligned}$$

Similarly, applying  $D_{-z}$  to the left-hand side of the second equation of (6.5),

$$\begin{aligned} D_{-z} K P_t(U_z^t) + D_{-z} V_z^t &= -K D_z P_t(U_z^t) + D_{-z} V_z^t \\ &= -K(P_t D_z(U_z^t) - iU_z^t(t)\delta_t(x)) + D_{-z} V_z^t. \end{aligned}$$

Applying  $D_{-z}$  to the right-hand side of the second equation of (6.5) and using (6.3),

$$\begin{aligned} D_{-z} g_z^t(x) &= \int_t^\infty i \frac{d}{dx} K(x+y) e^{izy} dy - z \int_t^\infty K(x+y) e^{izy} dy \\ &= -iK(x+t)e^{izt} + z \int_t^\infty K(x+y) e^{izy} dy - z \int_t^\infty K(x+y) e^{izy} dy. \end{aligned}$$

Combining the above calculations, we obtain the pair of equations

$$\begin{aligned} D_z U_z^t - K P_t D_{-z}(V_z^t) &= -iV_z^t(t)K_t, \\ -K P_t D_z(U_z^t) + D_{-z} V_z^t &= -i(e^{izt} + U_z^t(t))K_t. \end{aligned}$$

where we put  $K_t(x) = K(x+t)$ . By subtracting or adding these two equations, we obtain the pair of equations

$$\begin{aligned} D_z U_z^t - D_{-z} V_z^t + K P_t(D_z U_z^t - D_{-z} V_z^t) &= i(e^{izt} + U_z^t(t) - V_z^t(t))K_t, \\ D_z U_z^t + D_{-z} V_z^t - K P_t(D_z U_z^t + D_{-z} V_z^t) &= -i(e^{izt} + U_z^t(t) + V_z^t(t))K_t. \end{aligned}$$

This shows that  $D_z U_z^t - D_{-z} V_z^t$  and  $D_z U_z^t + D_{-z} V_z^t$  are solutions of

$$\phi_t^+ + K P_t \phi_t^+ = K_t, \quad \phi_t^- - K P_t \phi_t^- = K_t \quad (6.7)$$

up to constant multiplies, respectively (we made this repetition of (3.8) for convenience). On the other hand, equations (3.8) means that  $\phi_t^\pm(x) = \phi^\pm(t, x)$  are unique solutions of (6.7). Therefore, we get

$$\begin{aligned} D_z U_z^t - D_{-z} V_z^t &= i(e^{izt} + U_z^t(t) - V_z^t(t))\phi_t^+, \\ D_z U_z^t + D_{-z} V_z^t &= -i(e^{izt} + U_z^t(t) + V_z^t(t))\phi_t^-. \end{aligned}$$

Adding these equations,

$$D_z U_z^t = \frac{i}{2}(e^{izt} + U_z^t(t) - V_z^t(t))\phi_t^+ - \frac{i}{2}(e^{izt} + U_z^t(t) + V_z^t(t))\phi_t^-. \quad (6.8)$$

Applying  $K$  to both sides,

$$\begin{aligned} KP_t D_z U_z^t &= \frac{i}{2}(e^{izt} + U_z^t(t) - V_z^t(t))KP_t \phi_t^+ - \frac{i}{2}(e^{izt} + U_z^t(t) + V_z^t(t))KP_t \phi_t^- \\ &= \frac{i}{2}(e^{izt} + U_z^t(t) - V_z^t(t))(K_t - \phi_t^+) - \frac{i}{2}(e^{izt} + U_z^t(t) + V_z^t(t))(-K_t + \phi_t^-). \end{aligned}$$

Applying  $K$  to both sides again and using  $KK_t = \delta_t$  coming from  $K_t = K\delta_t$  and  $K^2 = \text{id}$ ,

$$\begin{aligned} P_t D_z U_z^t &= \frac{i}{2}(e^{izt} + U_z^t(t) - V_z^t(t))(\delta_t - K\phi_t^+) \\ &\quad - \frac{i}{2}(e^{izt} + U_z^t(t) + V_z^t(t))(-\delta_t + K\phi_t^-). \end{aligned} \quad (6.9)$$

From (6.6), we have

$$\begin{aligned} D_z Y_z^t(x) &= \mathbf{1}_{[t, \infty)}(x)(D_z U_z^t(x) + i(e^{izt} + U_z^t(t))\delta_t(x)) \\ &= (D_z U_z^t(x) - (P_t D_z U_z^t)(x) + i(e^{izt} + U_z^t(t))\delta_t(x)). \end{aligned} \quad (6.10)$$

Substituting (6.8) and (6.9) into the right-hand side of (6.10),

$$\begin{aligned} D_z Y_z^t(x) &= \frac{i}{2}(e^{izt} + U_z^t(t) - V_z^t(t))(\phi_t^+(x) + K\phi_t^+(x)) \\ &\quad - \frac{i}{2}(e^{izt} + U_z^t(t) + V_z^t(t))(\phi_t^-(x) - K\phi_t^-(x)). \end{aligned}$$

By putting

$$\psi_t(x) = \frac{1}{2}(\phi_t^+ - \phi_t^-)(x) + \frac{1}{2}K(\phi_t^+ + \phi_t^-)(x), \quad (6.11)$$

we obtain

$$D_z Y_z^t(x) = i(e^{izt} + U_z^t(t))\psi_t(x) - iV_z^t(t)K\psi_t(x). \quad (6.12)$$

Note that  $\psi_t$  is not a function but a tempered distribution on  $\mathbb{R}$ . In fact, we have  $K\phi_t^\pm = \delta_t \mp P_t \phi_t^\pm$  and so  $K(\phi_t^+ + \phi_t^-) = 2\delta_t - P_t \phi_t^+ + P_t \phi_t^-$  from (6.7). Thus

$$\psi_t(x) = \delta_t(x) + \frac{1}{2}\mathbf{1}_{[t, \infty)}(x)(\phi_t^+ - \phi_t^-)(x).$$

We have  $\phi_t^+ - \phi_t^- = -KP_t(\phi_t^+ + \phi_t^-) \in K(L^2(-\infty, t)) \subset L^2(\mathbb{R})$  from (6.7) again. Therefore, using  $K\phi_t^\pm = \delta_t \mp P_t \phi_t^\pm$  (and recall that  $\mathbf{1}_{[t, \infty)}e^{izx} \in L^2(\mathbb{R})$ ), we have

$$\begin{aligned} \int_t^\infty (\phi_t^+(x) - \phi_t^-(x))e^{izx} dx &= \int_{-\infty}^\infty (K\phi_t^+(x) - K\phi_t^-(x))g_z^t(x) dx \\ &= - \int_{-\infty}^t P_t(\phi_t^+(x) + \phi_t^-(x))P_t g_z^t(x) dx. \end{aligned}$$

We have  $P_t \phi_t^+ + K[t]\phi_t^+ = P_t K_t$  and  $P_t \phi_t^- - K[t]\phi_t^- = P_t K_t$  from (6.7). Therefore  $P_t(\phi_t^+ + \phi_t^-) + K[t](\phi_t^+ - \phi_t^-) = 2P_t K_t$  and  $P_t(\phi_t^+ - \phi_t^-) + K[t](\phi_t^+ + \phi_t^-) = 0$ , thus

$$(1 - K[t]^2)P_t(\phi_t^+ + \phi_t^-) = 2P_t K_t.$$

Using this,

$$\begin{aligned} \int_{-\infty}^t P_t \frac{\phi_t^+(x) + \phi_t^-(x)}{2} P_t g_z^t(x) dx &= \int_{-\infty}^t ((1 - K[t]^2)^{-1}P_t K_t)(x) P_t g_z^t(x) dx \\ &= \int_{-\infty}^t K(x+t)((1 - K[t]^2)^{-1}P_t g_z^t)(x) dx. \end{aligned}$$

The right-hand side may be written as  $\int_{-\infty}^t K(x+t)V_z^t(x) dx$ , since we have  $P_t V_z^t = (1 - K[t]^2)^{-1} P_t g_z^t$  by (6.5). Moreover,  $\int_{-\infty}^t K(x+t)V_z^t(x) dx = (KP_t V_z^t)(t) = -U_z^t(t)$  by using (6.5) again. Thus we obtain

$$(F\psi_t)(z) = e^{izt} + \frac{1}{2} \int_t^\infty (\phi_t^+(x) - \phi_t^-(x)) e^{izx} dx = e^{izt} + U_z^t(t).$$

In a similar way, we have

$$\frac{\phi_t^+ + \phi_t^-}{2} = K_t - KP_t \frac{\phi_t^+ - \phi_t^-}{2}$$

and

$$\begin{aligned} \int_t^\infty \frac{\phi_t^+(x) + \phi_t^-(x)}{2} e^{izx} dx &= \int_t^\infty K(x+t) e^{izt} dx + \int_t^\infty KP_t \frac{-\phi_t^+ + \phi_t^-}{2} \cdot e^{izt} dx \\ &= \int_t^\infty K(x+t) e^{izt} dx + \int_{-\infty}^t P_t \frac{-\phi_t^+(x) + \phi_t^-(x)}{2} \cdot P_t g_z^t(x) dx. \end{aligned}$$

The second integral on the right-hand side is calculated as

$$\begin{aligned} &\int_{-\infty}^t P_t \frac{-\phi_t^+(x) + \phi_t^-(x)}{2} \cdot P_t g_z^t(x) dx \\ &= \int_{-\infty}^t ((1 - K[t]^2)^{-1} K[t] K_t)(x) P_t g_z^t(x) dx \\ &= \int_{-\infty}^t K(x+t) ((1 - K[t]^2)^{-1} K[t] P_t g_z^t)(x) dx \\ &= - \int_{-\infty}^t K(x+t) (P_t U_z^t)(x) dx \\ &= -(KP_t U_z^t)(t) = V_z^t(t) - g_z^t(t) = V_z^t(t) - \int_t^\infty K(x+t) e^{izx} dx. \end{aligned}$$

Thus,  $\frac{1}{2} \int_t^\infty (\phi_t^+(x) + \phi_t^-(x)) e^{izx} dx = V_z^t(t)$ . On the other hand, applying  $K$  to (6.11),

$$K\psi_t(x) = \frac{1}{2} K(\phi_t^+ - \phi_t^-)(x) + \frac{1}{2} (\phi_t^+ + \phi_t^-)(x) = \frac{1}{2} \mathbf{1}_{[t, \infty)}(x) (\phi_t^+ + \phi_t^-)(x).$$

Therefore,  $(FK\psi_t)(z) = V_z^t(t)$ . Then, we obtain the reformulation of (6.12) as

$$D_z Y_z^t(x) = i(F\psi_t)(z) \psi_t(x) - i(FK\psi_t)(z) K\psi_t(x).$$

On the other hand, we have

$$(FD_z Y_z^t)(w) = (w+z)(FY_z^t)(w) = (w+z)(Y_z^t, Y_w^t)$$

for  $\Im(w) > 0$ . Hence, we obtain

$$\begin{aligned} (Y_z^t, Y_w^t) &= \int_t^\infty Y_z^t(x) Y_w^t(x) dx = \frac{(F\psi_t)(z)(F\psi_t)(w) - (FK\psi_t)(z)(FK\psi_t)(w)}{-i(w+z)} \\ &= \frac{(F\psi_t)(z)(F\psi_t)(w) - \Theta(z)(F\psi_t)(-z)\Theta(w)(F\psi_t)(-w)}{-i(w+z)}. \end{aligned}$$

We define  $\mathfrak{E}(t, z) := \mathfrak{A}(t, z) - i\mathfrak{B}(t, z)$  by using  $\mathfrak{A}(t, z)$  and  $\mathfrak{B}(t, z)$  defined in (3.27) (and extended to entire functions in Corollary 3.1). Then, we have

$$E(z)(F\psi_t)(z) = \mathfrak{A}(t, z) - i\mathfrak{B}(t, z) = \mathfrak{E}(t, z)$$

by (3.26), (3.28) and (3.29), and

$$\begin{aligned} E^\sharp(z)(F\psi_t)^\sharp(z) &= \mathfrak{A}(t, z) + i\mathfrak{B}(t, z) = \mathfrak{E}(t, -z) = E(-z)(F\psi_t)(-z); \\ \overline{\mathfrak{E}(t, z)} &= \overline{E(z)(F\psi_t)(z)} = \mathfrak{E}(t, -\bar{z}) = E(-\bar{z})(F\psi_t)(-\bar{z}). \end{aligned}$$

Hence we obtain

$$(Y_z^t, Y_w^t) = \frac{1}{E(z)E(w)} \frac{\mathfrak{E}(t, z)\mathfrak{E}(t, w) - \mathfrak{E}^\sharp(t, z)\mathfrak{E}^\sharp(t, w)}{-i(w + z)}. \quad (6.13)$$

This equation is proved under the restriction  $\Im(z) > c$  and  $\Im(w) > 0$ . However we have unique Hilbert space vectors  $Y_z^t \in \mathcal{V}_t$  for all  $z \in \mathbb{C}$  such that  $(Ff)(z) = (f, Y_z^t)$  for every  $f \in \mathcal{V}_t$ . Therefore, (6.13) holds throughout  $\mathbb{C} \times \mathbb{C}$  by analytic continuation.

By definition  $Y_z^t(x) = (1/E(z))\overline{X_z^t(x)}$ , we easily find  $\overline{Y_z^t} = Y_{-\bar{z}}^t$ . Therefore,

$$\langle X_z^t, X_w^t \rangle = \int_t^\infty X_z^t(x) \overline{X_w^t(x)} dx = \overline{E(z)}E(w) \int_t^\infty Y_{-\bar{z}}^t(x) Y_w^t(x) dx.$$

The integral on the right-hand side is calculated as

$$= \frac{\mathfrak{E}(t, -\bar{z})\mathfrak{E}(t, w) - \mathfrak{E}^\sharp(t, -\bar{z})\mathfrak{E}^\sharp(t, w)}{i(\bar{z} - w)} = 2\pi J(t; z, w)$$

by (6.13). Hence we complete the proof of formula (6.2).

Taking  $z = w \in \mathbb{C}_+$  in (6.2), we have

$$\frac{|E(t, z)|^2 - |E^\sharp(t, z)|}{4\pi\Im(z)} = J(t; z, z) = \frac{1}{2\pi} \|X_z^t\|^2 > 0.$$

Hence (1.3) holds. If  $E(t, z)$  has a real zero, the real zero is a common zero of  $A(t, z)$  and  $B(t, z)$ . On the other hand, as reviewed in Section 5.4, there exists a transfer matrix  $M_t(z)$  on  $\mathbb{C}$  such that  $(A(z) \ B(z)) = (A(t, z) \ B(t, z))M_t(z)$  and  $\det M_t(z) = 1$ . Therefore, if  $E(t, z)$  has a real zero,  $E(z) = A(z) - iB(z)$  also has a real zero. This contradicts  $E \in \mathbb{H}\mathbb{B}$ . Hence  $E(t, z)$  has no real zeros, and thus  $E(t, z) \in \mathbb{H}\mathbb{B}$ .

We have

$$\begin{aligned} \int_{\mathbb{R}} E(z)(Ff)(z) \frac{1}{2\pi} \overline{\langle X_w^t, X_z^t \rangle} \frac{1}{E(z)\overline{E(z)}} dz &= \frac{1}{2\pi} \int_{\mathbb{R}} (Ff)(z) \overline{\langle X_w^t, X_z^t \rangle} \frac{1}{\overline{E(z)}} dz \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} (Ff)(z) \overline{(FX_w^t)(z)} dz = \int_{\mathbb{R}} f(x) \overline{X_w^t(x)} dz = E(w)(Ff)(x) \end{aligned}$$

for every  $f \in \mathcal{V}_t$ . That is,  $J(t; z, w) = \frac{1}{2\pi} \langle X_z^t, X_w^t \rangle$  is the reproducing kernel of the de Branges subspace  $E(z)F(\mathcal{V}_t)$  of  $\mathcal{B}(E)$ . On the other hand,  $J(t; z, w)$  is the reproducing kernel of the de Branges subspace  $\mathcal{B}(E(t, z))$  of  $\mathcal{B}(E)$ . Hence  $E(z)F(\mathcal{V}_t) = \mathcal{B}(E(t, z))$ .  $\square$

**6.2. Proof of Theorem 2.3.** Suppose that  $J(t; z, w) \equiv 0$  for some  $t > 0$ . Then  $\mathcal{V}_t = \{0\}$ , since  $J(t; z, w)$  is the reproducing kernel of  $E(z)F(\mathcal{V}_t)$ . This contradicts Lemma 6.2, and thus  $J(t; z, w) \not\equiv 0$  for every  $t > 0$ .

We have

$$|J(t; z, w)| = |\langle X_z^t, X_w^t \rangle| \leq \|X_z^t\| \cdot \|X_w^t\|$$

for fixed  $z, w \in \mathbb{C}$  by Theorem 6.1. Therefore, for Theorem 2.3, it is sufficient to show that  $\|X_z^t\| \rightarrow 0$  as  $t \rightarrow \infty$  for fixed  $z \in \mathbb{C}$ . We have  $\|X_z^t\|^2 = \langle X_z^t, X_z^t \rangle = E(z)(FX_z^t)(z) = \langle X_z^t, X_z^0 \rangle$ , since  $\mathcal{V}_t \subset \mathcal{V}_0$ . Therefore,

$$\langle X_z^t, X_z^0 \rangle = \left| \int_t^\infty X_z^t(x) \overline{X_z^0(x)} dx \right| \leq \|X_z^t\| \left( \int_t^\infty |X_z^0(x)|^2 dx \right)^{1/2}$$

by the Cauchy–Schwarz inequality. Hence,  $\|X_z^t\|^2 \leq \int_t^\infty |X_z^0(x)|^2 dx$ . This show that  $\|X_z^t\| \rightarrow 0$  as  $t \rightarrow \infty$ , since  $X_z^0 \in L^2(0, \infty)$ . Theorem 2.3 is obtained by applying the above argument to  $E = E_L^{\omega, \nu}$  by Proposition 4.1.  $\square$

## 7. PROOF OF THEOREM 2.4

**7.1. Necessity.** If we take  $\nu > 1/(\omega d_L)$  for each  $\omega > 0$ ,  $E_L^{\omega, \nu}$  satisfies (K1)~(K4) by Proposition 4.1. In addition,  $\Theta_L^{\omega, \nu}$  is inner in  $\mathbb{C}_+$  for every  $(\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$  under GRH( $L$ ) by Proposition 4.3. Therefore, we obtain (K5) with  $\tau = \infty$  by Proposition 4.4. Thus we obtain (1), (2) and (3) of Theorem 2.4 by applying Theorem 3.1 to  $E = E_L^{\omega, \nu}$ . Moreover, we obtain (4) by Theorem 2.3.  $\square$

**7.2. Sufficiency.** By (1), (2) and (3), we obtain the Hamiltonian  $H_L^{\omega_n, \nu_n}$  on  $[0, \infty)$  having no  $H_L^{\omega_n, \nu_n}$ -indivisible intervals and the family of pairs  $(A_L^{\omega_n, \nu_n}(t, z), B_L^{\omega_n, \nu_n}(t, z))$ ,  $t \geq 0$ , solving the canonical system on  $[0, \infty)$  associated with  $H_L^{\omega_n, \nu_n}$  together with the initial condition

$$A_L^{\omega_n, \nu_n}(0, z) = A_L^{\omega_n, \nu_n}(z) \quad \text{and} \quad B_L^{\omega_n, \nu_n}(0, z) = B_L^{\omega_n, \nu_n}(z).$$

By Proposition 3.4, (4) implies that  $E_L^{\omega_n, \nu_n}(t, z) = A_L^{\omega_n, \nu_n}(t, z) - iB_L^{\omega_n, \nu_n}(t, z)$  belongs to  $\overline{\mathbb{H}\mathbb{B}}$  for every  $t \geq 0$ . In particular,

$$E_L^{\omega_n, \nu_n}(0, z) = A_L^{\omega_n, \nu_n}(z) - iB_L^{\omega_n, \nu_n}(z) = \xi_L(\tfrac{1}{2} + \omega_n - iz)^{\nu_n}$$

belongs to  $\overline{\mathbb{H}\mathbb{B}}$ . That is,  $|\xi_L(\tfrac{1}{2} - \omega_n - iz)^{\nu_n} / \xi_L(\tfrac{1}{2} + \omega_n - iz)^{\nu_n}| < 1$  if  $\Im(z) > 0$ . It implies that  $E_L^{\omega_n, 1}(z) = \xi_L(\tfrac{1}{2} + \omega_n - iz)$  belongs to  $\overline{\mathbb{H}\mathbb{B}}$ . In particular,  $\xi_L(\tfrac{1}{2} + \omega_n - iz)$  has no zeros in  $\mathbb{C}_+$  for every  $n$ . This implies that  $\xi_L(\tfrac{1}{2} - iz)$  has no zeros in  $\mathbb{C}_+$ . In fact, if  $\xi_L(\tfrac{1}{2} - iz)$  has a zero  $\gamma = u + iv \in \mathbb{C}_+$ , there exists  $n$  such that  $\omega_n < v$  and  $E_L^{\omega_n, 1}$  has a zero in  $\mathbb{C}_+$ , since  $\omega_n \rightarrow 0$  and

$$\xi_L(\tfrac{1}{2} - i(u + iv)) = \xi_L(\tfrac{1}{2} + \omega_n - i(u + i(v - \omega_n))).$$

This contradicts  $E_L^{\omega_n, 1} \in \overline{\mathbb{H}\mathbb{B}}$  for every  $n$ . The functional equation implies  $\xi_L(\tfrac{1}{2} - iz)$  has no zeros in  $\mathbb{C}_-$ . Hence, all zeros of  $\xi_L(\tfrac{1}{2} - iz)$  are real.  $\square$

8. SPECTRAL REALIZATION OF ZEROS OF  $A_L^{\omega, \nu}$  AND  $B_L^{\omega, \nu}$ 

In this part, we mention that the zeros of  $A_L^{\omega, \nu}$  and  $B_L^{\omega, \nu}$  can be regarded as eigenvalues of self-adjoint extensions of a differential operator for  $\omega > 1/2$  unconditionally and for  $0 < \omega \leq 1/2$  under GRH( $L$ ).

**8.1. Multiplication by the independent variable.** For  $E \in \mathbb{H}\mathbb{B}$ , the de Branges space  $\mathcal{B}(E)$  has the unbounded operator  $(M, \text{dom}(M))$  consisting of multiplication by the independent variable  $(Mf)(z) = zf(z)$  endowed with the natural domain  $\text{dom}(M) = \{f \in \mathcal{B}(E) \mid zf(z) \in \mathcal{B}(E)\}$ . The multiplication operator  $M$  is symmetric and closed, satisfies  $M(F^\sharp) = (MF)^\sharp$  for  $F \in \text{dom}(M)$  and has deficiency indices  $(1, 1)$  ([23, Proposition 4.2]). The operator  $M$  has no eigenvalues ([23, Corollary 4.3]).

In general,  $\text{dom}(M)$  has codimension at most one. Hereafter, we suppose that  $\text{dom}(M)$  has codimension zero, that is,  $\text{dom}(M)$  is dense in  $\mathcal{B}(E)$ . Then, all self-adjoint extensions  $M_\theta$  of  $M$  are parametrized by  $\theta \in [0, \pi)$  and their spectrum consists of eigenvalues only. The self-adjoint extension  $M_\theta$  is described as follows. We introduce

$$S_\theta(z) = e^{i\theta} E(z) - e^{-i\theta} E^\sharp(z)$$

for  $\theta \in [0, \pi)$ . The domain of  $M_\theta$  is defined by

$$\text{dom}(M_\theta) = \left\{ G_F(z) = \frac{S_\theta(w_0)F(z) - S_\theta(z)F(w_0)}{z - w_0} \mid F(z) \in \mathcal{B}(E) \right\}$$

and the operation is defined by

$$M_\theta G_F(z) = z G_F(z) + F(w_0) S_\theta(z),$$

where  $w_0$  is a fixed complex number with  $S_\theta(w_0) \neq 0$  and  $\text{dom}(\mathbf{M}_\theta)$  does not depend on the choice of  $w_0$ . The set

$$\left\{ F_{\theta,\gamma}(z) = \frac{S_\theta(z)}{z - \gamma} \mid S_\theta(\gamma) = 0 \right\}$$

forms an orthogonal basis of  $\mathcal{B}(E)$ , and each  $F_{\theta,\gamma}$  is an eigenfunction of  $\mathbf{M}_\theta$  with the eigenvalue  $\gamma$ :

$$\mathbf{M}_\theta F_{\theta,\gamma} = \gamma F_{\theta,\gamma}$$

([23, Proposition 6.1, Theorem 7.3]). We have  $S_{\pi/2}(z) = 2i A(z)$  and  $S_0(z) = -2i B(z)$  by definition. Therefore,  $\{A(z)/(z - \gamma) \mid A(\gamma) = 0\}$  and  $\{B(z)/(z - \gamma) \mid B(\gamma) = 0\}$  are orthogonal basis of  $\mathcal{B}(E)$  consisting of eigenfunctions of  $\mathbf{M}_{\pi/2}$  and  $\mathbf{M}_0$ , respectively.

**8.2. Transform to differential operators.** Considering the isometric isomorphism of the Hilbert spaces

$$\mathcal{B}(E) \rightarrow \mathcal{K}(\Theta) \rightarrow \mathcal{V}_0 \subset L^2(0, \infty),$$

we define the differential operator  $\mathbf{D}$  on  $\mathcal{V}_0$  by

$$\mathbf{D} = \mathbf{F}^{-1} \mathbf{M} \mathbf{F}, \quad \text{dom}(\mathbf{D}) = \mathbf{F}^{-1} \mathbf{M}_{\frac{1}{E}}(\text{dom}(\mathbf{M})) = \{f \in \mathcal{V}_0 \mid z(\mathbf{F}f)(z) \in \mathcal{K}(\Theta)\},$$

where  $\mathbf{M}_{\frac{1}{E}}$  is the operator of multiplication by  $1/E(z)$ . Then we have  $(\mathbf{D}f)(x) = i \frac{d}{dx} f(x)$  for  $f \in C^1(\mathbb{R}) \cap L^2(0, \infty)$ . All self-adjoint extensions of  $\mathbf{D}$  are given by

$$\mathbf{D}_\theta = \mathbf{F}^{-1} \mathbf{M}_\theta \mathbf{F}, \quad \text{dom}(\mathbf{D}_\theta) = \mathbf{F}^{-1} \mathbf{M}_{\frac{1}{E}}(\text{dom}(\mathbf{M}_\theta)), \quad \theta \in [0, \pi).$$

The set

$$\{f_{\theta,\gamma} = i e^{-i\theta} \mathbf{F}^{-1} \mathbf{M}_{\frac{1}{E}} F_{\theta,\gamma} \mid S_\theta(\gamma) = 0\}$$

forms an orthogonal basis of  $\mathcal{V}_0$  consisting of eigenfunctions  $f_{\theta,\gamma}$  of  $\mathbf{D}_\theta$  for eigenvalues  $\gamma$ , where  $i e^{-i\theta}$  is the constant for the simplicity of  $f_{\theta,\gamma}$ . We have

$$f_{\theta,\gamma}(x) = e^{-i\gamma x} \left( \mathbf{1}_{[0,\infty)}(x) - e^{-2i\theta} \int_0^x K(y) e^{i\gamma y} dy \right)$$

by the direct calculation of  $\mathbf{F}^{-1} \mathbf{M}_{\frac{1}{E}} F_{\theta,\gamma}$ . This formula suggests that the eigenfunction  $f_{\theta,\gamma}$  is an adjustment of the “eigenfunction”  $e^{-i\gamma x}$  of  $i \frac{d}{dx}$  in  $\mathcal{V}_0$ .

**Theorem 8.1.** *Let  $L \in \mathcal{S}_{\mathbb{R}}$ ,  $(\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$ . Suppose that  $E_L^{\omega,\nu} \in \overline{\mathbb{H}\mathbb{B}}$  so that the de Branges space  $\mathcal{B}(E_L^{\omega,\nu})$  is defined. Then  $\text{dom}(\mathbf{M})$  is dense in  $\mathcal{B}(E_L^{\omega,\nu})$ .*

*Proof.* Let  $E = E_L^{\omega,\nu}$ ,  $\Theta = E^\sharp/E$ . The domain of  $\mathbf{M}$  is not dense in  $\mathcal{B}(E)$  if and only if

$$\sum_{\gamma \in \mathbb{C}, \Theta(\gamma)=0} \Im(\gamma) < \infty. \quad (8.1)$$

by [9, Theorem 29] and [1, Theorem A and Corollary 2]. The condition (8.1) means that the zeros of  $E^\sharp$  appearing in the zeros of  $\Theta$  (all of them in  $\mathbb{C}_+$ ) are finitely many or tend to the real line sufficiently quickly (from the above) if  $\text{dom}(\mathbf{M})$  is not dense in  $\mathcal{B}(E)$ . If the number of such zeros is finite,  $\Theta$  is a rational function. But it contradicts (4.2) and (4.3). If  $E^\sharp$  has infinitely many zeros appearing in the zeros of  $\Theta$  and tend to the real line, the functional equation of  $\xi_L$  implies that there exists a zero of  $E^\sharp$  above the horizontal line  $\Im(z) = \omega$ . Hence, (8.1) is impossible.  $\square$

Theorem 8.1 suggests that the pair  $(\mathcal{V}_0, \mathbf{D}_\theta)$  is a Pólya-Hilbert space for  $A_L^{\omega,\nu}$  if  $\theta = \pi/2$  and for  $B_L^{\omega,\nu}$  if  $\theta = 0$ . Noting that  $A_L^{\omega,1} \rightarrow \xi_L$  as  $\omega \rightarrow 0$  if  $\epsilon_L = +1$  and  $B_L^{\omega,1} \rightarrow i\xi_L$  as  $\omega \rightarrow 0$  if  $\epsilon_L = -1$ , the family of pairs  $\{(\mathcal{V}_0^{\omega,\nu(\omega)}, \mathbf{D}_\theta^{\omega,\nu(\omega)})\}_{\omega>0}$  may be considered as a perturbation of the “genuine Pólya-Hilbert space” associated with  $E(z) = \xi_L(z) + \xi_L'(z)$ , where  $\nu(\omega) \rightarrow \infty$  as  $\omega \rightarrow 0$  under (2.8).



On the other hand,  $\mathcal{V}_0$  is isomorphic to the quotient space  $L^2(0, \infty)/K(L^2(-\infty, 0))$  by Lemma 6.1 and (5.1). This structure of  $\mathcal{V}_0$  is similar to Connes' suggestion for the Pólya-Hilbert space as explained below.

**8.3. Comparison with Connes' Pólya-Hilbert space.** Connes [14] suggests a candidate of the Pólya-Hilbert space by interpreting the critical zeros of the Riemann zeta function as an absorption spectrum as follows. Let  $S(\mathbb{R})_0$  be the subspace of the Schwartz space  $S(\mathbb{R})$  consisting of all even functions  $\phi \in S(\mathbb{R})$  satisfying  $\phi(0) = (F\phi)(0) = 0$ . For a function  $\phi \in S(\mathbb{R})_0$ , the function  $Z\phi$  on  $\mathbb{R}_+ = (0, \infty)$  is defined by  $(Z\phi)(y) = y^{1/2} \sum_{n=1}^{\infty} \phi(ny)$ . Then  $Z\phi$  is of rapid decay as  $y \rightarrow +0$  and  $y \rightarrow +\infty$ . In particular,  $Z(S(\mathbb{R})_0) \subset L^2(\mathbb{R}_+, dy/y)$ . Then the “orthogonal complement”  $L^2(\mathbb{R}_+, dy/y) \ominus Z(S(\mathbb{R})_0)$  is spanned by generalized eigenfunctions  $y^{-i\gamma}(\log y)^k$ ,  $0 \leq k < m_\gamma$ , of the differential operator  $iyd/dy$  attached to the critical zeros  $1/2 + i\gamma$  of the Riemann zeta function with multiplicity  $m_\gamma$ . That is,  $(iyd/dy, L^2(\mathbb{R}_+, dy/y) \ominus Z(S(\mathbb{R})_0))$  forms a “Pólya-Hilbert space”. The differential operator  $iyd/dy$  may be regarded as the shift of the Hamiltonian  $(1/2)(y[-i\hbar d/dy] + [-i\hbar d/dy]y) = -i\hbar(yd/dy + 1/2)$  of the Berry-Keating model [2].

Rigorously, the above argument does not make sense, since  $y^{-i\gamma}(\log y)^k$  are not members of  $L^2(\mathbb{R}_+, dy/y)$  and  $L^2(\mathbb{R}_+, dy/y) = \overline{Z(S(\mathbb{R})_0)}$ . However, the above naive idea is justified by several manners ([14] and R. Meyer [32, 33]), but some nice property such as the self-adjointness of the operator, the spectral realization of zeros, the Hilbert space structure is lost by known justification.

Contrast with justifications so far, the family  $\{(\mathcal{V}_0^{\omega, \nu(\omega)}, D_{\pi/2}^{\omega, \nu(\omega)})\}_{\omega > 0}$  justifies Connes' idea preserving the self-adjointness of the operator, the spectral realization of zeros and the Hilbert space structure by considering the perturbation family  $A_\xi^{\omega, \nu}$  of  $\xi$ .

Major objects of the above naive model of Connes' idea correspond to objects attached to  $(\mathcal{V}_0, D_\theta)$  as follows under the changing of variables  $y = e^x$ :

$$\begin{aligned} L^2(\mathbb{R}_+^\times, dy/y) &\Leftrightarrow L^2(0, \infty) \\ Z(S(\mathbb{R})_0) &\Leftrightarrow K(L^2(-\infty, 0)) \\ \text{Mellin}(Z(S(\mathbb{R})_0)) = \zeta(\tfrac{1}{2} + s) \text{Mellin}(S(\mathbb{R})_0) &\Leftrightarrow F(K(L^2(-\infty, 0))) = \Theta(z)F(L^2(0, \infty)) \\ iyd/dy &\Leftrightarrow D_\theta \approx i \frac{d}{dx}, \end{aligned}$$

where “Mellin” means the usual Mellin transform and  $\approx$  means “is equal up to domain”.

## 9. COMPLEMENTARY RESULTS AND DIGRESSIVE TOPICS

**9.1. A variant of Theorem 2.4.** By Proposition 4.3 and argument in Section 7, we obtain the following variant of Theorem 2.4.

**Theorem 9.1.** *Let  $L \in \mathcal{S}_\mathbb{R}$  and  $0 < \omega_0 < 1/2$ . Then  $L(s) \neq 0$  for  $\Re(s) > \frac{1}{2} + \omega_0$  if and only if there exists a sequence  $(\omega_n, \nu_n) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$ ,  $n \geq 1$ , such that*

- (1)  $\omega_m < \omega_n$  if  $m > n$  and  $\omega_n \rightarrow \omega_0$  as  $n \rightarrow \infty$ ,
- (2)  $\nu_n \omega_n d_L > 1$ ,
- (3)  $H_L^{\omega_n, \nu_n}(t)$  extends to a Hamiltonian on  $[0, \infty)$ , that is,  $\det(1 \pm K_L^{\omega_n, \nu_n}[t]) \neq 0$  for every  $t \geq 0$ , and
- (4)  $\lim_{t \rightarrow \infty} J_L^{\omega_n, \nu_n}(t; z, z) = 0$  for every fixed  $z \in \mathbb{C}_+$ .

**9.2. Inner property and isometry.** We prove that the converse of Lemma 3.2 holds in the following sense. For  $\Theta = E^\sharp/E$ , the multiplication  $F \mapsto \Theta F$  defines a map from  $L^2(\mathbb{R})$  into  $L^2(\mathbb{R})$  by (3.1). We denote it also by  $\Theta$  if no confusion arises, and define

$$K_\Theta = F^{-1}\Theta FJ : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),$$

where  $(Jf)(x) = f(-x)$ . If  $\Theta$  is an inner function in  $\mathbb{C}_+$ , images  $\Theta(H^2)$  and  $K_\Theta(L^2(\mathbb{R}))$  are subspaces of  $H^2$  and  $L^2(\mathbb{R})$ , respectively. Obviously the map  $K_\Theta$  is related to the function  $K$  by (K2).

**Lemma 9.1.** *Let  $E$  be an entire function satisfying (K1)~(K3). Suppose that  $\Theta = E^\sharp/E$  is uniformly bounded on  $\Im(z) \geq c'$  for some  $c' > 0$ . Then  $\Theta H^2 \subset H^2$  implies that  $\Theta$  is inner in  $\mathbb{C}_+$ .*

*Proof.* We know (3.1), and the assumption implies that  $\Theta$  has no poles in  $\mathbb{C}_+$ . Hence, by applying the Phragmén-Lindelöf convexity principle to  $\Theta$  in the strip  $0 \leq \Im(z) \leq c'$ , we find that  $\Theta$  is bounded on  $0 \leq \Im(z) \leq c'$ . Therefore  $\Theta$  is a bounded analytic function in  $\mathbb{C}_+$  satisfying (3.1). This is the definition of an inner function in  $\mathbb{C}_+$ .  $\square$

**Theorem 9.2.** *Let  $E$  be as in Lemma 9.1. Then  $\Theta = E^\sharp/E$  is a meromorphic inner function in  $\mathbb{C}_+$  if and only if one of the following conditions holds:*

- (1)  $Kf$  is defined as a function on  $\mathbb{R}$  and  $K_\Theta f = Kf$  for every  $f \in L^2(-\infty, 0)$ ;
- (2)  $K_\Theta f$  vanishes on  $(-\infty, 0)$  for every  $f \in L^2(-\infty, 0)$ ;
- (3)  $Kf$  is defined as a function of  $L^2(\mathbb{R})$  for every  $f \in L^2(-\infty, 0)$ ,

where we understand  $Kf$  by the integral of (3.2). In particular,  $\Theta = E^\sharp/E$  is a meromorphic inner function in  $\mathbb{C}_+$  if and only if  $f \mapsto Kf$  defines an isometry on  $L^2(\mathbb{R})$ .

The above equivalence can be applied to  $E = E_L^{\omega, \nu}$  for  $L \in \mathcal{S}_\mathbb{R}$  and  $(\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}$ .

*Proof.* The last line of Theorem 9.2 will be established if the other assertions are proved, since  $E_L^{\omega, \nu}$  satisfies (K1)~(K3) by Proposition 4.1 and  $\Theta_L^{\omega, \nu}$  is uniformly bounded on  $\Im(z) \geq 1/2 + \omega + \delta$  by Lemma 4.2. Therefore, it is sufficient to prove the following three assertions: i) condition (1) is equivalent that  $\Theta$  is inner in  $\mathbb{C}_+$ , ii) condition (2) implies that  $\Theta$  is inner in  $\mathbb{C}_+$ , and iii) condition (3) implies that  $\Theta$  is inner in  $\mathbb{C}_+$ , since (1) implies (2) and (3) by (K3) and definition of  $K_\Theta$ , respectively.

i) Suppose that  $\Theta$  is inner in  $\mathbb{C}_+$ . Then  $\Theta(z)F(-z) \in H^2$  for every  $F \in \bar{H}^2$ . Thus the inverse Fourier transform along the line  $\Im(z) = a$

$$(K_\Theta f)(x) = \frac{1}{2\pi} \int_{\Im(z)=a} \Theta(z)F(-z)e^{-izx} dz$$

is independent of  $a > 0$ , and belongs to  $L^2(0, \infty)$ , where  $f = F^{-1}F \in L^2(-\infty, 0)$  and the integral converges in the sense of  $L^2$ . On the other hand, we have

$$(Kf)(x) = \frac{1}{2\pi} \int_{\Im(z)=a} \Theta(z)F(-z)e^{-izx} dz$$

for  $a > c$  by (3.3) and [43, Theorem 65], where the integral converges also in the sense of  $L^2$ . Comparing these two formula for large  $a$ , we obtain (1).

Conversely, suppose that (1) holds. Write  $g = K_\Theta f = Kf$  for arbitrary fixed  $f \in L^2(-\infty, 0)$ . Then  $g$  belongs to  $L^2(\mathbb{R})$ , since  $K_\Theta$  maps  $L^2(\mathbb{R})$  to  $L^2(\mathbb{R})$  by definition. In addition,  $g$  has a support in  $[0, \infty)$ , since both  $K(x)$  and  $f(-x)$  have support in  $[0, \infty)$ . Therefore  $g$  belongs to  $L^2(0, \infty)$ . Because  $f$  was arbitrary, we have  $\Theta H^2 \subset H^2$ . Hence  $\Theta$  is inner in  $\mathbb{C}_+$  by Lemma 9.1.

ii) Suppose that (2) holds. Then it implies  $K_\Theta(L^2(-\infty, 0)) \subset L^2(0, \infty)$ , since  $K_\Theta$  maps  $L^2(\mathbb{R})$  into  $L^2(\mathbb{R})$  by its definition. It means  $\Theta H^2 \subset H^2$  by definition of  $K_\Theta$ . Hence  $\Theta$  is inner in  $\mathbb{C}_+$  by Lemma 9.1.

iii) Suppose that (3) holds. Then  $Kf$  belongs to  $L^2(0, \infty)$  for every  $f \in L^2(-\infty, 0)$ , since  $Kf$  has a support in  $[0, \infty)$  for  $f \in L^2(-\infty, 0)$  by its definition. Therefore  $FKf \in H^2$ . Additionally, we suppose that  $f$  belongs to the dense subset  $L^1(-\infty, 0) \cap L^2(-\infty, 0)$ . Then  $(FKf)(z) = \Theta(z)(Ff)(-z)$  for  $\Im(z) > 1/2 + \omega$  by [43, Theorem 44]. Therefore

$$K_\Theta(L^1(-\infty, 0) \cap L^2(-\infty, 0)) \subset L^2(0, \infty).$$

This implies  $K_\Theta L^2(-\infty, 0) \subset L^2(0, \infty)$ , since  $K_\Theta$  is continuous by its definition. Therefore  $\Theta H^2 \subset H^2$  by definition of  $K_\Theta$ , and hence  $\Theta$  is inner in  $\mathbb{C}_+$  by Lemma 9.1.  $\square$

**9.3. A sufficient condition for the invertibility of  $1 \pm K[t]$ .** In general, for an entire function  $E$  satisfying (K1)~(K3), the existence of  $\tau > 0$  in (K5) is not obvious even if we assume that  $\Theta = E^\sharp/E$  is inner in  $\mathbb{C}_+$ . We may need an additional condition for  $\Theta$  to conclude the existence of  $\tau > 0$  in (K5) as well as the role of Lemma 4.3 in the proof of Proposition 4.4. A simple sufficient condition for (K5) with  $\tau = \infty$  is as follows.

**Proposition 9.1.** *Let  $E$  be an entire function satisfying (K1)~(K3). In addition, we suppose that  $\Theta$  is inner in  $\mathbb{C}_+$  and there is no entire functions  $F$  and  $G$  of exponential type such that  $\Theta = G/F$ . Then both  $\pm 1$  are not eigenvalues of  $K[t]$  for every  $t > 0$ .*

*Proof.* We suppose that  $K[t]f = \pm f$  for some  $0 \neq f \in L^2(-\infty, t)$ . Then we find that  $g := Kf$  has a support in  $[-t, t]$  in a way similar to the proof of Proposition 4.4. By taking the Fourier transform, we have  $(Fg)(z) = \Theta(z)(Ff)(-z)$ . Thus  $\Theta(z) = (Fg)(z)/(Ff)(-z)$ . This is a contradiction, since  $Ff$  and  $Fg$  are entire functions of exponential type by the Paley–Wiener theorem.  $\square$

**9.4. Dirac and Schrödinger equations.** The canonical system in Theorem 3.1 is equivalent to the following Dirac equation. We put

$$\mathfrak{X}_z(t) = \begin{bmatrix} \mathfrak{A}(t, z) \\ \mathfrak{B}(t, z) \end{bmatrix}, \quad \mathfrak{X}^0(t) = \begin{bmatrix} m(t)^{-1} & 0 \\ 0 & m(t) \end{bmatrix}, \quad X_z(t) = \begin{bmatrix} A(t, z) \\ B(t, z) \end{bmatrix}.$$

Then,  $\mathfrak{X}_z(t) = \mathfrak{X}^0(t)X_z(t)$  by (3.38), and  $\mathfrak{X}_z(t)$  solves the Dirac equation

$$J\mathfrak{X}'_z + Q\mathfrak{X}_z = z\mathfrak{X}_z, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & \mu \\ \mu & 0 \end{bmatrix} \quad (9.1)$$

from (3.27) and (3.30). The matrix-valued function  $\mathfrak{X}^0(t)$  is the unique matrix solution of (9.1) for  $z = 0$  satisfying  $\det \mathfrak{X}^0(t) = 1$  and  $\mathfrak{X}^0(0) = I_2$  by (3.32). Therefore, substituting  $\mathfrak{X}_z(t) = \mathfrak{X}^0(t)X_z(t)$  in (9.1), and then multiplying the transpose  ${}^t\mathfrak{X}^0(t)$  in both sides, we obtain the canonical system in Theorem 3.1 associated with the Hamiltonian  $H(t)$  of (3.33) and  $H(t) = {}^t\mathfrak{X}^0(t)\mathfrak{X}^0(t)$ . Obviously, the above derivation is reversible.

The Dirac equation (9.1) derives a pair of Schrödinger equations endowed with potentials having simple formulas. Differentiating both sides of (9.1), and then substituting  $\mathfrak{X}'_z = J(Q - zI_2)\mathfrak{X}_z$  obtained from (9.1) and  $J^2 = -I_2$ , we have  $J\mathfrak{X}''_z + Q'\mathfrak{X}_z + (Q - zI_2)J(Q - zI_2)\mathfrak{X}_z = 0$ . Using  ${}^tAJA = (\det A)J$  and  $\det(Q - zI_2) = z^2 - \mu^2$ , we get  $J\mathfrak{X}''_z + (Q' + (z^2 - \mu^2)J)\mathfrak{X}_z = 0$ . Multiplying by  $J$ , we obtain the pair of Schrödinger equations

$$\left( -\frac{d^2}{dt^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} q^+(t) & 0 \\ 0 & q^-(t) \end{bmatrix} \right) \mathfrak{X}_z = z^2 \mathfrak{X}_z,$$

where

$$q^\pm(t) = \mu(t)^2 \mp \mu'(t) = \frac{1}{4} \left( \frac{\gamma'(t)}{\gamma(t)} \right)^2 \mp \frac{1}{2} \left( \frac{\gamma'(t)}{\gamma(t)} \right)'$$

if the right-hand sides are defined. Both elements  $\mathfrak{A}(t, z)$  and  $\mathfrak{B}(t, z)$  of  $\mathfrak{X}_z$  belong to  $L^2(0, \tau)$  as a function of  $t$  for fixed  $z \in \mathbb{C}$  if  $\lim_{t \rightarrow \tau} J(t; z, z) = 0$  by (3.42).

**Proposition 9.2.** *We have*

$$q^\pm(t) = -2 \frac{d^2}{dt^2} \log \det(1 \pm K[t]) = \mp 2 \frac{d}{dt} \phi^\pm(t, t). \quad (9.2)$$

*Proof.* We prove (9.2) in a way similar to the argument in [13, Section 6]. Let  $W(t; x, y) = \int_{-\infty}^t K(x+z)K(z+y) dz$  be the kernel of the operator  $W_t := K[t]^2$  on  $L^2(-\infty, t)$ . Using the translation operator  $T_t : L^2(-\infty, 0) \rightarrow L^2(-\infty, t)$ , we define

$$w_t := T_t^{-1}W_tT_t : L^2(-\infty, 0) \rightarrow L^2(-\infty, 0).$$

The operator  $w_t$  has the kernel

$$w(t; x, y) = \int_{-\infty}^0 K(x + z + 2t)K(z + y + 2t) dz.$$

We have

$$\frac{d}{dt}w(t; x, y) = 2K(x + 2t)K(y + 2t)$$

by a direct calculation. Therefore, for  $\Phi \in L^2(-\infty, 0)$ , we obtain

$$\left( \frac{d}{dt}w_t \Phi \right)(x) = 2 \cdot \int_{-\infty}^0 K(y + 2t)\Phi(y) dy \cdot K(x + 2t).$$

That is,  $\frac{d}{dt}w_t$  is a rank one operator on  $L^2(-\infty, 0)$  with range  $\mathbb{C} K(x + 2t) \cdot \mathbf{1}_{(-\infty, 0)}(x)$ . The well-known formula

$$\frac{d}{dt} \log \det(1 - w_t) = -\text{Tr} \left( (1 - w_t)^{-1} \frac{d}{dt} w_t \right).$$

shows that the function  $2((1 - w_t)^{-1}K(\cdot + 2t))(x) \cdot \mathbf{1}_{(-\infty, 0)}(x)$  is a eigenvector of the rank one operator  $(1 - w_t)^{-1} \frac{d}{dt} w_t$  having the eigenvalue  $2 \int_{-\infty}^0 K(y + 2t)((1 - w_t)^{-1}K(\cdot + 2t))(y) dy$ . Going back to  $(-\infty, t)$ , the eigenvalue is

$$2 \int_{-\infty}^t K(y + t)((1 - W_t)^{-1}K(\cdot + t))(y) dy.$$

On the other hand, we have

$$\int_{-\infty}^t \phi^+(t, x)\phi^-(t, x) dx = \int_{-\infty}^t ((1 - W_t)^{-1}K(\cdot + t))(x) \cdot K(x + t) dx.$$

by  $P_t \phi_t^\pm = (1 \pm K[t])^{-1} P_t K_t$ . Hence we obtain

$$\frac{d}{dt} \log \det(1 - W_t) = -2 \int_{-\infty}^t \phi^+(t, x)\phi^-(t, x) dx.$$

Using (3.24), we have

$$\frac{d}{dt} \int_{-\infty}^t \phi^+(t, x)\phi^-(t, x) dx = \frac{1}{2}(\phi^+(t, t) + \phi^-(t, t))^2 = \frac{1}{2}\mu(t)^2.$$

Therefore, we obtain

$$\mu(t)^2 = -\frac{d^2}{dt^2} \log \det(1 - W_t) = -\frac{d^2}{dt^2} \log \det(1 + K[t]) - \frac{d^2}{dt^2} \log \det(1 - K[t]).$$

On the other hand, we have

$$\mu(t) = \phi^+(t, t) + \phi^-(t, t) = \frac{d}{dt} \log \det(1 + K_t) - \frac{d}{dt} \log \det(1 - K_t),$$

by (3.34), and hence

$$q^\pm(t) = \mu(t)^2 \mp \mu'(t) = -2 \frac{d^2}{dt^2} \log \det(1 \pm K[t]) = \mp 2 \frac{d}{dt} \phi^\pm(t, t).$$

This implies (9.2). □

**9.5. Expected partial differential equations for Fourier integrands.** It had been shown in Section 3 that entire function  $E$  satisfying (K1)~(K5) can be recovered as (3.41) from Fourier transforms (3.40) of functions  $F(t, x) = m(t)\mathfrak{F}(t, x)$  and  $G(t, x) = m(t)^{-1}\mathfrak{G}(t, x)$  satisfy the partial differential equation (3.37). Suppose that the equality

$$\lim_{x \rightarrow +\infty} \frac{\mathfrak{F}(t, x)}{\mathfrak{G}(t, x)} = -1 \quad (9.3)$$

holds. Then we have

$$\gamma(t) = m(t)^2 = \lim_{x \rightarrow +\infty} \frac{F(t, -x) + F(t, x)}{G(t, -x) - G(t, x)}.$$

Therefore,  $(F(t, x), G(t, x))$  is characterized as the unique solution of the following “boundary-value problem”:

$$\begin{cases} \frac{\partial}{\partial t} F(t, x) + \gamma(t) \frac{\partial}{\partial x} G(t, x) = 0, \\ \frac{\partial}{\partial t} G(t, x) + \frac{1}{\gamma(t)} \frac{\partial}{\partial x} F(t, x) = 0, \\ \gamma(t) = \lim_{x \rightarrow +\infty} \frac{F(t, -x) + F(t, x)}{G(t, -x) - G(t, x)}, \\ F(0, x) = \frac{1}{2}(\varrho(x) + \varrho(-x)), \quad G(0, x) = \frac{1}{2}(\varrho(x) - \varrho(-x)). \end{cases} \quad (9.4)$$

We should remark that  $\gamma(t)$  is not a given function in (9.4) different from the initial-value problem (3.37) with  $F(0, x) = \frac{1}{2}(\varrho(x) + \varrho(-x))$  and  $G(0, x) = \frac{1}{2}(\varrho(x) - \varrho(-x))$ . In other words,  $\gamma(t)$  is determined from the third line of (9.4) if it is solved.

Therefore,  $\gamma_L^{\omega, \nu}$  is described in terms of the solution of (9.4) if (9.3) holds for  $E = E_L^{\omega, \nu}$ . Such description of  $\gamma_L^{\omega, \nu}$  may help us to extend  $\gamma_L^{\omega, \nu}$  to a wider interval without GRH( $L$ ). Unfortunately, we do not know whether (9.3) holds for  $E = E_L^{\omega, \nu}$  at present, but it is expected by considering and comparing with the case of  $\varrho \in C_c^\infty(\mathbb{R})$ .

**9.6. Hilbert spaces isomorphic to de Branges subspaces.** Let  $E$  be an entire function satisfying (K1)~(K5). Then  $\gamma(t) = m(t)^2$  in (3.33) is a continuous positive-valued function on  $[0, \tau)$ . For an interval  $I \subset [0, \tau)$ , we define

$$\mathcal{H}(\gamma)_I = \left\{ u = (u_1, u_2) : I \rightarrow \mathbb{C}^2 \left| \int_I (\gamma(t)^{-1} |u_1(t)|^2 + \gamma(t) |u_2(t)|^2) dt < \infty \right. \right\}.$$

Then  $\mathcal{H}(\gamma)_I$  forms a Hilbert space under the inner product

$$\langle u, v \rangle = \int_I (\gamma(t)^{-1} u_1(t) \overline{v_1(t)} + \gamma(t) u_2(t) \overline{v_2(t)}) dt.$$

If  $I \subset J$ , we regard  $\mathcal{H}(\gamma)_I \subset \mathcal{H}(\gamma)_J$  by the extension by zero.

For the time being, we suppose that  $E \in \mathbb{HB}$  and  $\tau = \infty$  in addition. Then, the de Branges space  $\mathcal{B}(E)$  is defined, and the Hilbert space  $\mathcal{H}(\gamma)_{[0, \infty)}$  is isometrically isomorphic to  $\mathcal{B}(E)$  as a Hilbert space by the *de Branges transform*

$$(\mathbb{T}u)(z) = \frac{1}{\pi} \int_0^\infty (u_1(t) \gamma(t)^{-1} A(t, z) + u_2(t) \gamma(t) B(t, z)) dt.$$

Moreover,  $\mathcal{H}(\gamma)_{[\tau, \infty)}$  is isometrically isomorphic to  $\mathcal{B}(E(\tau, z))$  via  $\mathbb{T}$  for every  $\tau \geq 0$  by [9, Theorem 44] with a changing of variables. Therefore, we obtain the isomorphism

$$\mathcal{H}(\gamma)_{[\tau, \infty)} \xrightarrow[\mathbb{T}]{\sim} \mathcal{B}(E(\tau, z)) \xrightarrow[\mathbb{F}^{-1} \mathbb{M}_{\frac{1}{E}}]{\sim} \mathcal{V}_\tau$$

for every  $\tau \geq 0$  by Theorem 6.1. We easily find that  $\mathbf{1}_{[\tau, \infty)}(t)(A(t, \bar{w}), B(t, \bar{w}))$  belongs to  $\mathcal{H}(\gamma)_{[\tau, \infty)}$  as a function of  $t$  and it is transformed into  $J(\tau; w, z)$  by the proof of

Proposition 3.4. From the direct and orthogonal decomposition  $\mathcal{H}(\gamma)_{[0,\infty)} = \mathcal{H}(\gamma)_{[0,\tau)} \oplus \mathcal{H}(\gamma)_{[\tau,\infty)}$ , we obtain the isomorphism

$$\mathcal{H}(\gamma)_{[0,\tau)} \simeq \mathcal{V}_0 \oplus \mathcal{V}_\tau$$

given by  $u \mapsto F^{-1}M_{\frac{1}{E}}Tu$ . The image of  $u = (u_1, u_2) \in \mathcal{H}(\gamma)_{[0,\tau)}$  is calculated as

$$\begin{aligned} (F^{-1}M_{\frac{1}{E}}Tu)(x) &= \frac{1}{2\pi} \int_0^\tau u_1(t)(\phi_t^+(x) + (\mathcal{K}\phi_t^+)(x)) \frac{1}{m(t)} dt \\ &\quad + \frac{1}{2\pi i} \int_0^\tau u_2(t)(\phi_t^-(x) - (\mathcal{K}\phi_t^-)(x)) m(t) dt, \end{aligned}$$

since

$$\frac{A(t, z)}{E(z)} = F \left( m(t) \frac{1}{2} (\phi_t^+ + \mathcal{K}\phi_t^+) \right) (z), \quad \frac{B(t, z)}{E(z)} = F \left( \frac{1}{m(t)} \frac{1}{2i} (\phi_t^- - \mathcal{K}\phi_t^-) \right) (z)$$

by definitions (3.29) and (3.38). Using  $\mathcal{K}\phi_t^+ = \delta_t - \phi_t^+ \mathbf{1}_{(-\infty, t)}$  and  $\mathcal{K}\phi_t^- = \delta_t + \phi_t^- \mathbf{1}_{(-\infty, t)}$  obtained from (6.7), we have

$$\begin{aligned} 2\pi(F^{-1}M_{\frac{1}{E}}Tu)(x) &= \mathbf{1}_{[0,\tau)}(x) \left[ \frac{1}{m(x)} u_1(x) + im(x)u_2(x) \right. \\ &\quad \left. + \int_0^x \left( \frac{1}{m(t)} u_1(t)\phi_t^+(x) - im(t)u_2(t)\phi_t^-(x) \right) dt \right] \\ &\quad + \mathbf{1}_{[\tau,\infty)}(x) \int_0^\tau \left( \frac{1}{m(t)} u_1(t)\phi_t^+(x) - im(t)u_2(t)\phi_t^-(x) \right) dt. \end{aligned}$$

Using the right-hand side, the image  $F^{-1}M_{\frac{1}{E}}T(\mathcal{H}(\gamma)_{[0,\tau')})$  is defined for all  $\tau' \leq \tau$  without the assumption that  $E \in \mathbb{H}\mathbb{B}$  as well as the space  $\mathcal{H}(\gamma)_{[0,\tau')}$  if we have (K1)~(K5). Therefore, two spaces of functions  $\mathcal{H}(\gamma_L^{\omega,\nu})_{[0,\tau)}$  and  $F^{-1}M_{\frac{1}{E}}T(\mathcal{H}(\gamma_L^{\omega,\nu})_{[0,\tau)})$  are defined without GRH( $L$ ) for sufficiently small  $\tau > 0$  by the above argument and results in Section 4. These spaces are isometrically isomorphic to  $\mathcal{V}_0 \oplus \mathcal{V}_\tau$  under GRH( $L$ ) as above, but it seems that it is difficult to prove such isomorphism without GRH( $L$ ).

**9.7. Necessary conditions for  $\lim_{t \rightarrow \infty} J(t; z, w) = 0$ .** Under the conditions of Proposition 3.4, we suppose that  $\lim_{t \rightarrow \infty} J(t; z, w) = 0$  for any  $z, w \in \mathbb{C}$  and  $E(0) = A(0) \neq 0$ . Then, we obtain  $\int_0^\infty (1/\gamma(t)) dt < \infty$  and  $\lim_{t \rightarrow \infty} B(t, z) = 0$  for any  $z \in \mathbb{C}$ . In fact, we have  $A(t, 0) = A(0)$  for any  $t \geq 0$  by (3.40) and (3.37), and

$$J(0; z, w) = \frac{1}{\pi} \int_0^\infty \overline{A(t, z)} A(t, w) \frac{1}{\gamma(t)} dt + \frac{1}{\pi} \int_0^\infty \overline{B(t, z)} B(t, w) \gamma(t) dt.$$

from (3.42) and  $\lim_{t \rightarrow \infty} J(t; z, w) = 0$ . Hence, we obtain  $J(0; 0, 0) = \frac{A(0)^2}{\pi} \int_0^\infty \frac{1}{\gamma(t)} dt$ . On the other hand,  $J(t; 0, z) = \overline{A(t, 0)} B(t, z) / (\pi z) = A(0) B(t, z) / (\pi z)$  by definition, since  $B(t, z)$  is odd. Hence  $\lim_{t \rightarrow \infty} B(t, z) = 0$ .

Note that  $\int_0^\infty (1/\gamma(t)) dt < \infty$  is a part of the sufficient condition in [9, Theorem 41] for the existence of the solution of the canonical system for  $H(t) = \text{diag}(1/\gamma(t), \gamma(t))$ .

## 10. MISCELLANEOUS REMARKS

(1) Concerning the size of Dirichlet coefficients  $a_L(n)$ , the polynomial bound  $|a_L(n)| \ll n^A$  for some  $A \geq 0$  is enough to prove Theorems 2.1, 2.2 and 2.3. In other words, the Ramanujan conjecture (S4) is not necessary to prove these theorems. Therefore, the method of Sections 3 and 4 about the construction of  $\text{Inv}(E_L^{\omega,\nu})^\flat$  can be applicable to more general  $L$ -functions, in particular, to  $L$ -functions associated to self-dual irreducible cuspidal automorphic representations of  $GL_n(\mathbb{A}_{\mathbb{Q}})$  with unitary central characters.

(2) In contrast with the Ramanujan conjecture (S4), the Euler product (S5) is essential to the construction of  $\text{Inv}(E_L^{\omega,\nu})^\flat$ . In fact, the explicit formula of the kernel  $K_L^{\omega,\nu}$  coming



from (S5) was critical to prove that  $E_L^{\omega,\nu}$  satisfies (K2) and (K3). It seems that it is not easy even to prove that  $K_L^{\omega,\nu}$  is a function if we do not have (S5).

It is an interesting problem to extend the construction of  $\text{Inv}(E_L^{\omega,\nu})^b$  to the class of  $L$ -data which is an axiomatic framework for  $L$ -functions introduced by A. Booker [3]. Superficially, Booker's  $L$ -datum does not require the Euler product, but it is based on the Weil explicit formula of  $L$ -functions in the Selberg class. Roughly, the Weil explicit formula is a result of (S3) and (S5), but the theory of  $L$ -data suggests that the Weil explicit formula is more essential than (S3) and (S5).

(3) By the Euler product (S5),  $L \in \mathcal{S}$  is expressed as a product of local  $p$ -factors  $L_p$ , and often, there exists polynomial  $F_p$  of degree at most  $d_L$  for each prime  $p$  such that  $L_p(s) = 1/F_p(p^{-s})$ . The Ramanujan conjecture (S4) is understood as the analogue of the Riemann hypothesis for  $F_p(p^{-s})$ . The Hamiltonian  $H_{L,p}$  attached to  $F_p(p^{-s})$  was constructed in [41] by using a way analogous to the method in Section 3 if  $F_p$  is a real self-reciprocal polynomial (for details, see [41, Section 1, Section 7.6]). It is an interesting problem to find a relation among the perturbation family of global Hamiltonians  $H_L^{\omega,\nu}$ , the family of local Hamiltonians  $H_{L,p}$  and the conjectural Hamiltonian  $H_L$  corresponding to  $E(z) = \xi_L(s) + \xi'_L(s)$ .

(4) The method of [41] for the construction of  $\text{Inv}(E)$  (not  $\text{Inv}(E)^b$ ) for exponential polynomials  $E$  is useful to observe  $H_L^{\omega,\nu}$  by numerical calculation of computer for concrete given  $L \in \mathcal{S}_{\mathbb{R}}$ , because an entire function satisfies (K1) is approximated by exponential polynomials by approximating the Fourier integral by Riemann sums (cf. the final part of the introduction of [41]).

(5) A sharp estimate of  $K_L^{\omega,\nu}(x)$  for large  $x > 0$  is not necessary to prove the main results of this paper. In fact, we do not know the role of the behavior of  $K_L^{\omega,\nu}(x)$  at  $x = +\infty$  in the equivalent condition of Theorem 2.4.

(6) If we replace the condition (4) of Theorem 2.4 by

$$(4') \quad E_L^{\omega_n, \nu_n}(0) \neq 0 \text{ and } \lim_{t \rightarrow \infty} (A_L^{\omega_n, \nu_n}(t, z), B_L^{\omega_n, \nu_n}(t, z)) = (E_L^{\omega_n, \nu_n}(0), 0),$$

we obtain a sufficient condition for  $\text{GRH}(L)$ , since (4)' implies (4). It is ideal if this is also a necessary condition, but we have no plausible evidence to support the necessity of (4)'. On the contrary, it is not clear whether  $\lim_{t \rightarrow +\infty} A_L^{\omega, \nu}(t, z)$  defines a functions of  $z$  contrast with the fact  $\lim_{t \rightarrow +\infty} B_L^{\omega, \nu}(t, z) = 0$  under  $\omega > 1/2$  or  $\text{GRH}(L)$  (see Section 9.7). The limit behavior may be related with the arithmetic properties of  $L(s)$  in a deep level, because we need information of all  $q_L^{\omega, \nu}(n)$ 's to understand it differ from the situation that we need only finitely many  $q_L^{\omega, \nu}(n)$ 's to understand  $K_L^{\omega, \nu}[t]$  for a finite range of  $t \in \mathbb{R}$ . We do not touch this problem further in this paper.

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